Lecture 22: Triple Integrals

22.1 Triple integrals

Similar to the case of double integrals, if $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous on a closed box

$$B = [a, b] \times [c, d] \times [e, f],$$

then

$$\int\int\int_B f(x, y, z)dV = \int_a^b \int_c^d \int_e^f f(x, y, z)dzdydx.$$

Of course, the integral may be evaluated in any order. Moreover, if D is a region of the form

$$D = \{(x, y, z) : a \le x \le b, g(x) \le y \le h(x), u(x, y) \le z \le v(x, y)\},$$

where g, h, u, and v are all continuous functions, then we have

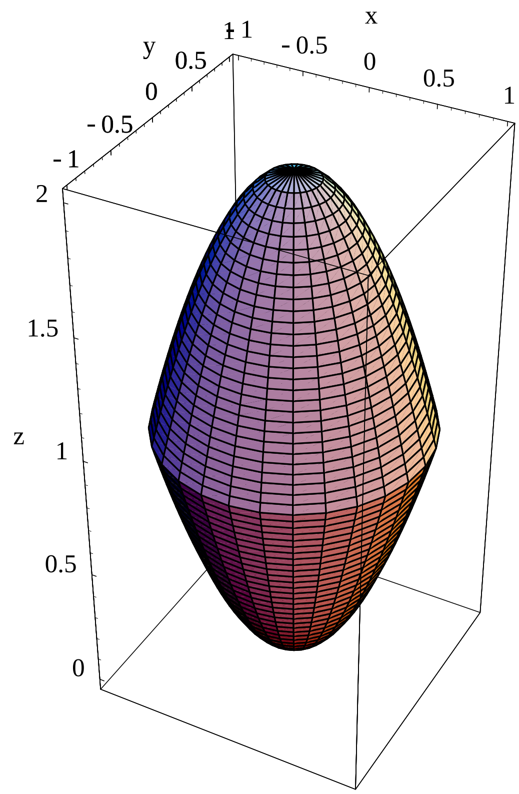
$$\int\int\int_D f(x, y, z)dV = \int_a^b \int_{g(x)}^{h(x)} \int_{u(x,y)}^{v(x,y)} f(x, y, z)dzdydx,$$

with similar results for analogously defined regions. Moreover, if V is the volume of D, then

$$V = \int\int\int_D dV.$$

Example If $B = [-1, 2] \times [0, 2] \times [0, 1]$, then

$$\begin{aligned}
\int\int\int_B xyzdV &= \int_{-1}^2 \int_0^2 \int_0^1 xyzdzdydx \\
&= \int_{-1}^2 \int_0^2 \frac{1}{2}xyz^2\Big|_0^1 dydx \\
&= \int_{-1}^2 \int_0^2 \frac{1}{2}xydydx \\
&= \int_{-1}^2 \frac{1}{4}xy^2\Big|_0^2 dx \\
&= \int_{-1}^2 xdx \\
&= \frac{1}{2}x^2\Big|_{-1}^2 \\
&= 2 - \frac{1}{2} \\
&= \frac{3}{2}.
\end{aligned}$$



Region bounded by $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$

Example If V is the volume of the region D in $\mathbb{R}^3$ bounded by the surfaces $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$, then

$$V = \int\int\int_D dV.$$

Now the two surfaces intersect when

$$2 - x^2 - y^2 = x^2 + y^2,$$

that is, when

$$x^2 + y^2 = 1.$$

Hence the surfaces intersect in the circle $x^2 + y^2 = 1$ at a height $z = 1$ above the xy-plane. Thus

$$D = \{(x, y, z) : -1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, x^2 + y^2 \leq z \leq 2 - x^2 - y^2\}.$$

Hence

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} dzdydx$$

$$
\begin{aligned}
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - 2x^2 - 2y^2) dy dx \\
&= \int_{0}^{1} \int_{0}^{2\pi} 2(1 - r^2) r d\theta dr \\
&= \int_{0}^{1} \int_{0}^{2\pi} 2(r - r^3) d\theta dr \\
&= 4\pi \int_{0}^{1} (r - r^3) dr \\
&= 4\pi \left(\frac{1}{2} r^2 \Big|_0^1 - \frac{1}{4} r^4 \Big|_0^1 \right) \\
&= 4\pi \left(\frac{1}{2} - \frac{1}{4} \right) \\
&= \pi.
\end{aligned}
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