Lecture 21: Polar Coordinates

21.1 Polar coordinates for the plane

We may describe a point P, other than the origin, in the plane by specifying the distance r from P to the origin and the angle θ between the line from the origin to P and the horizontal axis (measured in the counterclockwise direction). If P has Cartesian coordinates (x, y), then

$$r = \sqrt{x^2 + y^2}$$

and, if $x \neq 0$,

$$\tan(\theta) = \frac{y}{x}.$$

Conversely, if P has polar coordinates (r, θ), then

$$x = r\cos(\theta)$$

and

$$y = r\sin(\theta).$$

Example If P has Cartesian coordinates $(2, 2)$, then P has polar coordinates

$$r = 2\sqrt{2}$$

and

$$\theta = \frac{\pi}{4}.$$

Note that we could also use

$$\theta = \frac{9\pi}{4}.$$

Example If P has polar coordinates $\left(4, \frac{2\pi}{3}\right)$, then P has Cartesian coordinates

$$x = -2$$

and

$$y = 2\sqrt{3}.$$

Example Let

$$D = \{(x, y) : x^2 + y^2 \leq 4\}.$$

In polar coordinates, we have

$$D = \{(r, \theta) : r \leq 2, 0 \leq \theta \leq 2\pi\}.$$

21.2 Double integrals in polar coordinates

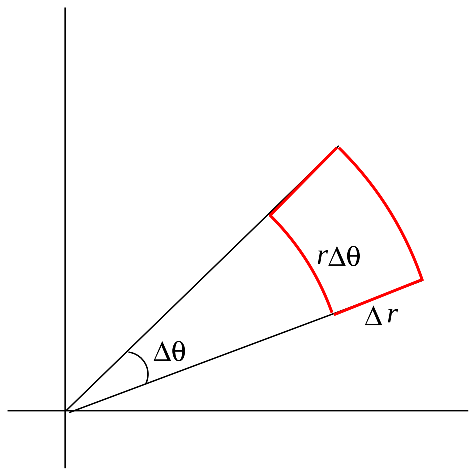
Note: If, working in polar coordinates, A is the area of the sector

$$S = \{(s, \alpha) : r \leq s \leq r + \Delta r, \theta \leq \alpha \leq \theta + \Delta\theta\},$$

then

$$A \approx r\Delta\theta\Delta r,$$

not $\Delta r\Delta\theta$ (see the figure below).



Area of a sector of an annulus is approximately $r\Delta\theta\Delta r$

Because of this, if f is continuous on a region in the plane, described by D in Cartesian coordinates and by T in polar coordinates, then

$$\int\int_D f(x, y)dxdy = \int\int_T f(r\cos(\theta), r\sin(\theta))rdrd\theta.$$

Example If V is the volume of the region beneath the surface $z = 16 - x^2 - y^2$ and above the xy-plane, then

$$\begin{aligned} V &= \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}(16 - x^2 - y^2)dydx \\ &= \int_0^{2\pi}\int_0^4 (16 - r^2)rdrd\theta \\ &= \int_0^{2\pi}\left(8r^2 - \frac{1}{4}r^4\right)\Big|_0^4 d\theta \\ &= \int_0^{2\pi}(128 - 64)d\theta \\ &= 64\theta\Big|_0^{2\pi} \\ &= 128\pi. \end{aligned}$$

Example If $D = \{(x,y) : 1 \le x^2 + y^2 \le 4, 0 \le x \le 2, 0 \le y \le 2\}$, then

$$\begin{aligned}\int\int_D e^{-\sqrt{x^2+y^2}} dA &= \int_1^2 \int_0^{\frac{\pi}{2}} re^{-r} d\theta dr \\ &= \frac{\pi}{2}\int_1^2 re^{-r} dr \\ &= \frac{\pi}{2}\left(-re^{-r}\Big|_1^2 + \int_1^2 e^{-r} dr\right) \\ &= \frac{\pi}{2}\left(-2e^{-2} + e^{-1} - e^{-r}\Big|_1^2\right) \\ &= \frac{\pi}{2}(2e^{-1} - 3e^{-2}).\end{aligned}$$

Example Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx.$$

Then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx dy.$$

Changing to polar coordinates, we have

$$\begin{aligned}I^2 &= \int_0^{\infty}\int_0^{2\pi} re^{-\frac{r^2}{2}} d\theta dr \\ &= 2\pi \int_0^{\infty} re^{-\frac{r^2}{2}} dr \\ &= 2\pi \lim_{b\to\infty} \int_0^b re^{-\frac{r^2}{2}} dr \\ &= -2\pi \lim_{b\to\infty} e^{-\frac{r^2}{2}}\Big|_0^b \\ &= -2\pi \lim_{b\to\infty}\left(e^{-\frac{b^2}{2}} - 1\right) \\ &= 2\pi.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}.$$