Lecture 20: Type I and Type II Regions

20.1 General regions

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on a bounded region D in $\mathbb{R}^n$. To define the integral of f over D, we let R be a rectangle which contains D, define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x_1, x_2, \ldots, x_n) = \begin{cases} f(x_1, x_2, \ldots, x_n), & \text{if } (x_1, x_2, \ldots, x_n) \text{ is in } D, \\ 0, & \text{if } (x_1, x_2, \ldots, x_n) \text{ is not in } D, \end{cases}$$

and let

$$\int\int_D f(x,y)dA = \int\int_R g(x,y)dA.$$

20.2 Regions of type I and type II

Suppose $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are both continuous on $[a, b]$ with $g(x) \leq h(x)$ for all x in $[a, b]$. We call the region

$$D = \{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$$

a region of *type* I. If f is continuous on D, then we have

$$\int\int_D f(x,y)dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx.$$

Suppose $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are both continuous on $[a, b]$ with $g(y) \leq h(y)$ for all y in $[c, d]$. We call the region

$$D = \{(x, y) : c \leq y \leq d, g(y) \leq x \leq h(y)\}$$

a region of *type* II. If f is continuous on D, then we have

$$\int\int_D f(x,y)dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)dxdy.$$

Example Let D be the triangle in the plane with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$. Then have

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x\},$$

so

$$\begin{aligned}\int\int_D xydA &= \int_0^1 \int_0^{1-x} xydydx \\ &= \int_0^1 \frac{1}{2}xy^2\Big|_0^{1-x} dx \\ &= \int_0^1 \frac{1}{2}x(1-x)^2 dx \\ &= \int_0^1 \frac{1}{2}(x - 2x^2 + x^3)dx \\ &= \frac{1}{4}x^2\Big|_0^1 - \frac{1}{3}x^3\Big|_0^1 + \frac{1}{8}x^4\Big|_0^1 \\ &= \frac{1}{4} - \frac{1}{3} + \frac{1}{8} \\ &= \frac{1}{24}.\end{aligned}$$

Example Sometimes changing the order of integration can help in the evaluation of an integral. For example,

$$\begin{aligned}\int_0^1 \int_x^1 e^{y^2} dydx = \int_0^1 \int_0^y e^{y^2} dxdy \\ &= \int_0^1 ye^{y^2} dx \\ &= \frac{1}{2}e^{y^2}\Big|_0^1 \\ &= \frac{1}{2}(e-1).\end{aligned}$$

Example If V is the volume of the region in $\mathbb{R}^3$ bounded by $z = 16 - x^2 - y^2$ and the xy-plane, then

$$V = \int\int_D (16 - x^2 - y^2)dA$$

where

$$D = \{(x,y) : x^2 + y^2 \leq 16\}.$$

Hence

$$\begin{aligned}V &= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (16 - x^2 - y^2)dydx \\ &= \int_{-4}^4 \left(16y - x^2y - \frac{1}{3}y^3\right)\Big|_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx \\ &= \int_{-4}^4 \left(32\sqrt{16-x^2} - 2x^2\sqrt{16-x^2} - \frac{2}{3}(16-x^2)^{\frac{3}{2}}\right) dx\end{aligned}$$

$$= 32\int_{-4}^{4}\sqrt{16-x^2}dx - 2\int_{-4}^{4}x^2\sqrt{16-x^2}dx - \frac{2}{3}\int_{-4}^{4}(16-x^2)^{\frac{3}{2}}dx.$$

If we let

$$x = 4\sin(u)$$
$$dx = 4\cos(u),$$

then

$$\begin{aligned}\int_{-4}^{4}\sqrt{16-x^2}dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{16-16\sin^2(u)}\;4\cos(u)du\\ &= 16\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2(u)du\\ &= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos(2u))du\\ &= 8u\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 4\sin(2u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= 8\pi,\end{aligned}$$

$$\begin{aligned}\int_{-4}^{4}x^2\sqrt{16-x^2}dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}256\sin^2(u)\cos^2(u)du\\ &= 256\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{2}\sin(2u)\right)^2du\\ &= 64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2(2u)du\\ &= 32\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos(4u))du\\ &= 32u\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 8\sin(4u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= 32\pi,\end{aligned}$$

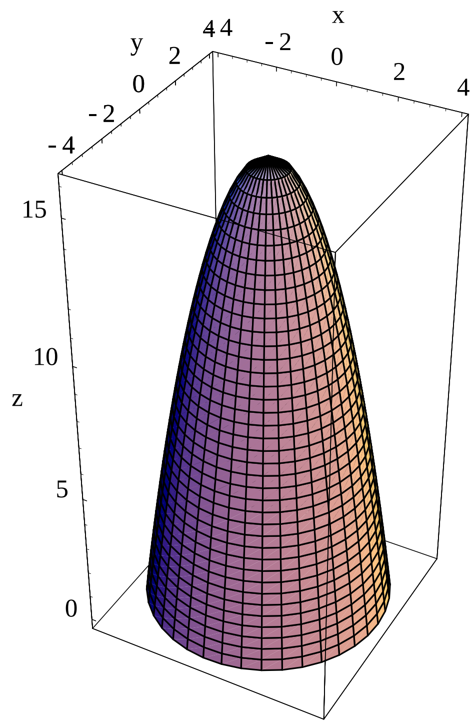
and

$$\begin{aligned}\int_{-4}^{4}(16-x^2)^{\frac{3}{2}}dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}256\cos^4(u)du\\ &= 256\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1}{2}(1+\cos(2u))\right)^2du\\ &= 64\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos(2u)+\cos^2(2u))du\\ &= 64u\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 64\sin(2u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 32\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos(4u))du\end{aligned}$$

$$= 64\pi + 32u\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 8\sin(4u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= 96\pi.$$

Hence

$$V = 256\pi - 64\pi - 64\pi = 128\pi.$$



Region between the surface $z = 16 - x^2 - y^2$ and the xy-plane

20.3 Properties of double integrals

If f and g are both integrable on D, then

$$\int\int_D (f(x,y) + g(x,y))dA = \int\int_D f(x,y)dA + \int\int_D g(x,y)dA.$$

If f is integrable on D and c is a scalar,

$$\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA.$$

If f and g are both integrable on D and $f(x,y) \leq g(x,y)$ for all (x,y) in D, then

$$\int\int_D f(x,y)dA \leq \int\int_D g(x,y)dA.$$

For any scalar c,

$$\int\int_D c\,dA = c \times \text{ (area of } D).$$

In particular,

$$\int\int_D dA = \text{ area of } D.$$

If $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$m \times \text{ (area of } D) \leq \int\int_D f(x, y)dA \leq M \times \text{ (area of } D).$$

If $D = D_1 \cup D_2$, where D_1 and D_2 are disjoint, and f is integrable on D, D_1, and D_2, then

$$\int\int_D f(x, y)dA = \int\int_{D_1} f(x, y)dA + \int\int_{D_2} f(x, y)dA.$$