Lecture 2: Vectors

2.1 Points and vectors

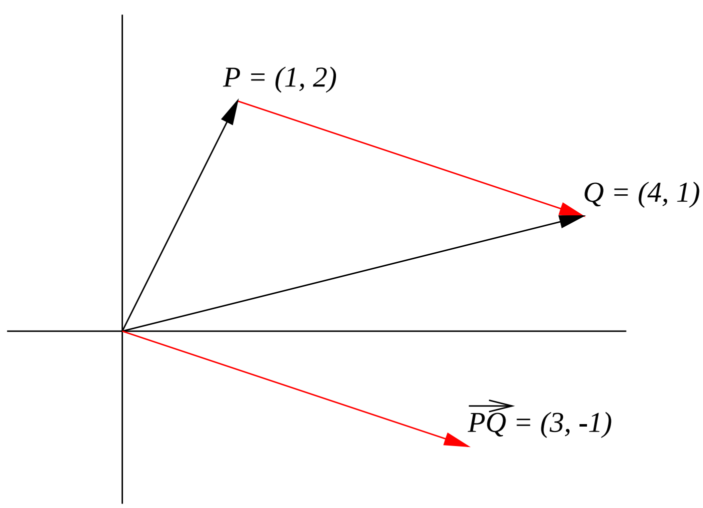
When we think of a point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ as representing both a distance and a direction from the origin $\mathbf{0}$, we may call $\mathbf{x}$ a *vector*. Geometrically, it is often convenient to picture $\mathbf{x}$ as an arrow. In this context, a real number is often called a *scalar*. The language derives from physics where vectors are used to represent quantities which have both magnitude and direction and scalars are used for quantities which have magnitude only.

We will often find it convenient to picture a vector as an arrow with its tail located at a point other than the origin. In particular, if we speak of a vector from the point $P = (a_1, a_2, \ldots, a_n)$ to the point $Q = (b_1, b_2, \ldots, b_n)$, then we are speaking of the vector

$$\overrightarrow{PQ} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n).$$

Example The vector from $P = (1, 2)$ to $Q = (4, 1)$ in the plane is

$$\overrightarrow{PQ} = (3, -1).$$



The vector $\overrightarrow{PQ} = (3, -1)$

The *length*, or *magnitude*, of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, denoted either $|\mathbf{x}|$ or $\|\mathbf{x}\|$, is the distance of the tip of the vector from the origin, that is,

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example If $\mathbf{x} = (1, 2, 3)$, then

$$|\mathbf{x}| = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

2.2 The algebra of vectors

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbb{R}^n$, then we define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n),$$

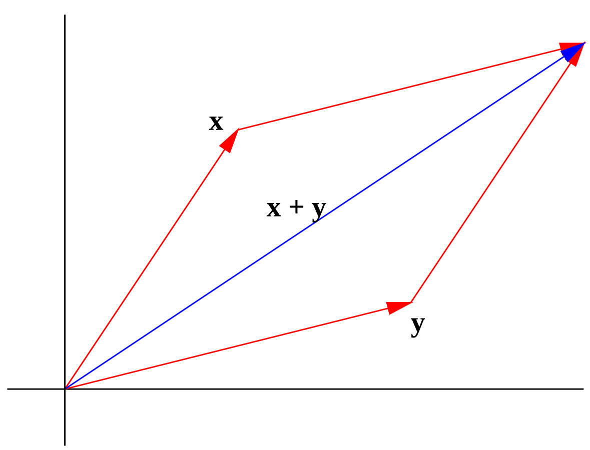
and, for any scalar c,

$$c\mathbf{x} = (cx_1, cx_2, \ldots, cx_n).$$

Example If $\mathbf{x} = (2, 3)$ and $\mathbf{y} = (4, 1)$, then

$$\mathbf{x} + \mathbf{y} = (6, 4).$$

Note that, geometrically, $\mathbf{x} + \mathbf{y}$ is the diagonal of the parallelogram with vertices at $(0, 0)$, $(2, 3)$, $(6, 4)$, and $(4, 1)$.

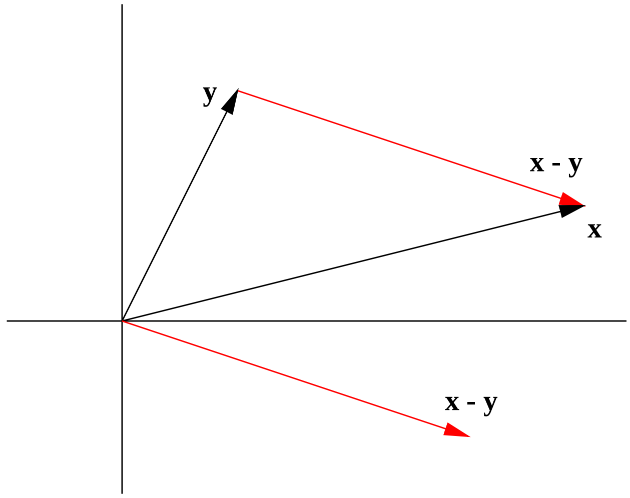


The sum of $\mathbf{x} = (2, 3)$ and $\mathbf{y} = (4, 1)$

In general, $\mathbf{x} + \mathbf{y}$ is the diagonal of the parallelogram with adjacent sides $\mathbf{x}$ and $\mathbf{y}$.

Example If $\mathbf{x} = (4, 1)$ and $\mathbf{y} = (1, 2)$, then

$$\mathbf{x} - \mathbf{y} = (3, -1).$$



The difference of $\mathbf{x} = (4, 1)$ and $\mathbf{y} = (1, 2)$

Note that, geometrically, $\mathbf{x} - \mathbf{y}$ is the vector from the tip of $\mathbf{y}$ to the tip of $\mathbf{x}$. Hence $\mathbf{x} - \mathbf{y}$ is the same as $\overrightarrow{PQ}$ of our previous example where P and Q were the points $(1, 2)$ and $(4, 1)$.

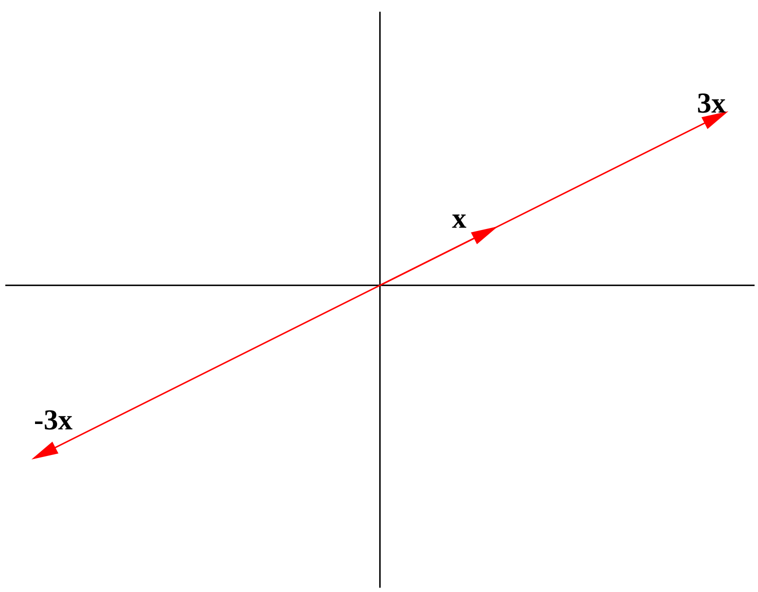
In general, $\mathbf{x} - \mathbf{y}$ is the vector from the tip of $\mathbf{y}$ to the tip of $\mathbf{x}$.

Example If $\mathbf{x} = (2, 1)$, then $3\mathbf{x} = (6, 3)$ and $-3\mathbf{x} = (-6, -3)$. Note that

$$|3\mathbf{x}| = \sqrt{45} = 3\sqrt{5} = 3|\mathbf{x}|$$

and

$$|-3\mathbf{x}| = \sqrt{45} = 3\sqrt{5} = 3|\mathbf{x}|.$$



Scalar multiples of $\mathbf{x} = (2, 1)$

Note that, in general, for any scalar c and vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$,

$$\begin{aligned} |c\mathbf{x}| &= |(cx_1, cx_2, \ldots, cx_n)| \\ &= \sqrt{c^2x_1^2 + c^2x_2^2 + \cdots + c^2x_n^2} \\ &= |c|\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\ &= |c||\mathbf{x}|. \end{aligned}$$

Geometrically, $c\mathbf{x}$ is the vector $\mathbf{x}$ stretched by the factor $|c|$, with direction reversed if $c < 0$.

We say that nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are *parallel* if $\mathbf{x} = c\mathbf{y}$ for some scalar c. Hence, for example, the vectors $(2, 1)$ and $(-6, -3)$ of the previous example are parallel.

2.3 Unit vectors

Note that if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is any vector, $\mathbf{x} \neq \mathbf{0}$, then

$$\left|\frac{1}{|\mathbf{x}|}\mathbf{x}\right| = \frac{1}{|\mathbf{x}|}|\mathbf{x}| = 1.$$

That is,

$$\frac{1}{|\mathbf{x}|}\mathbf{x}$$

is a *unit vector*, that is, a vector of length one, with the same direction as $\mathbf{x}$.

Example If $\mathbf{x} = (3, 2)$, then

$$\mathbf{u} = \frac{1}{\sqrt{13}}(3, 2)$$

is a unit vector with the same direction as $\mathbf{x}$.

The unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ are the *standard basis vectors* for $\mathbb{R}^3$. Note that if $\mathbf{a} = (a, b, c)$ is a vector in $\mathbb{R}^3$, then we may write

$$\mathbf{a} = (a, 0, 0) + (0, b, 0) + (0, 0, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Example $(1, 2, 3) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

More generally, the vectors

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, 0, \ldots, 0), \\ \mathbf{e}_2 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ \mathbf{e}_n &= (0, 0, 0, \ldots, 1) \end{aligned}$$

are the *standard basis vectors* for $\mathbb{R}^n$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector in $\mathbb{R}^n$, then we may write

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$