Lecture 19: Iterated Integrals

19.1 Another look at volume

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on the closed rectangle $R = [a, b] \times [c, d]$ with $f(x, y) \geq 0$ for all (x, y) in R. Let V be the volume of the region

$$S = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, 0 \leq z \leq f(x, y)\}.$$

Then we know that

$$V = \int\int_R f(x, y)dA.$$

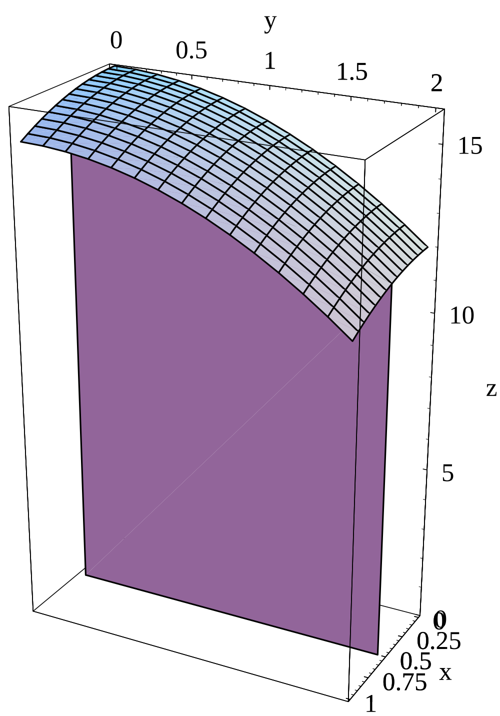
For $a \leq c \leq b$, let

$$A(c) = \int_c^d f(c, y)dy.$$

Then $A(c)$ is the area of a slice of S parallel to the yz-plane, namely, the intersection of the plane $x = c$ with S. Thus if we divide $[a, b]$ into n intervals of equal length $\Delta x = \frac{b-a}{n}$ and choose points c_1, c_2, $\ldots$, c_n with c_i in the ith interval, we will have

$$V \approx \sum_{i=1}^{n} A(c_i)\Delta x$$

and



A slice of the region under the graph of $y = 16 - x^2 - y^2$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\Delta x = \int_a^b A(x)dx = \int_a^b \int_c^d f(x,y)dydx.$$

We call the latter integral an *iterated integral*.

Example Let

$$S = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 2, 0 \le 16 - x^2 - y^2\}.$$

If V is the volume of S, then

$$\begin{aligned} V &= \int_0^1 \int_0^2 (16 - x^2 - y^2)dydx \\ &= \int_0^1 \left(16y - x^2 y - \frac{1}{3}y^3\right)\Big|_0^2 dx \\ &= \int_0^1 \left(32 - 2x^2 - \frac{8}{3}\right) dx \\ &= \int_0^2 \left(\frac{88}{3} - 2x^2\right) dx \\ &= \left(\frac{88}{3}x - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{88}{3} - \frac{2}{3} \\ &= \frac{86}{3}. \end{aligned}$$

Note that since we also know that

$$V = \int\int_R (16 - x^2 - y^2)dA,$$

we must have

$$\int\int_R (16 - x^2 - y^2)dA = \int_0^1 \int_0^2 (16 - x^2 - y^2)dydx.$$

19.2 Fubini's theorem

Fubini's Theorem If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $R = [a,b] \times [c,d]$, then

$$\int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

Example If $R = [0,1] \times [-1,1]$, then

$$\begin{aligned}\int\int_R xe^{xy}dA &= \int_{-1}^{1}\int_{0}^{1} xe^{xy}dxdy \\ &= \int_{0}^{1}\int_{-1}^{1} xe^{xy}dydx \\ &= \int_{0}^{1} e^{xy}\Big|_{-1}^{1}dx \\ &= \int_{0}^{1}(e^x - e^{-x})dx \\ &= e^x\Big|_0^1 + e^{-x}\Big|_0^1 \\ &= (e-1) + (e^{-1}-1) \\ &= e + \frac{1}{e} - 2.\end{aligned}$$

Note that the integration could have been done in either order, but the order we chose was a little simpler (avoiding an integration by parts).