Lecture 18: Multiple Integrals

18.1 Cartesian products

Notation: If A and B are sets, then we let

$$A \times B = \{(a, b) : a \text{ is in } A \text{ and } b \text{ is in } B\}.$$

We call $A \times B$ the *Cartesian product* of A and B. In particular, for closed intervals $[a, b]$ and $[c, d]$,

$$[a, b] \times [c, d] = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$$

is a closed rectangle in $\mathbb{R}^2$. If $[e, f]$ is another closed interval, then

$$[a, b] \times [c, d] \times [e, f] = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, e \leq z \leq d\}$$

is a closed box in $\mathbb{R}^3$.

18.2 Multiple integrals

Recall: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. To define the definite integral of f on a closed interval $[a, b]$, we first divide $[a, b]$ in to n intervals of equal length $\Delta x = \frac{b-a}{n}$. We then choose points c_1, c_2, $\ldots$, c_n, with c_i in the ith interval. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

a limit which is guaranteed to exist by the continuity of f.

Now suppose $f : R^n \to \mathbb{R}$ is continuous on an n-dimensional rectangle

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

If we divide each interval $[a_i, b_i]$ into m_i intervals of equal length

$$\Delta x_i = \frac{b_i - a_i}{m_i}$$

and let c_{ij} be a point in the the jth interval, then we define

$$\int \cdots \int_R f(x_1, x_2, \ldots, x_n)dV = \lim_{m_n\to\infty,\ldots,m_1\to\infty} \sum_{i_n=1}^{m_n} \cdots \sum_{i_1=1}^{m_1} f(c_{1i_1}, \ldots, c_{ni_n})\Delta x_1 \cdots \Delta x_n.$$

If $n = 2$, we will use dA instead of dV.

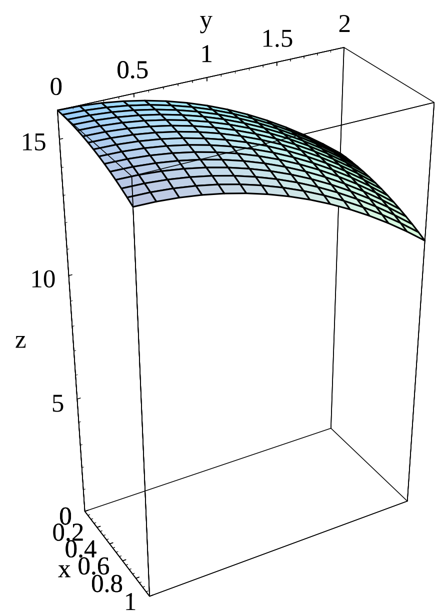
18.3 Double integrals

If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $R = [a, b] \times [c, d]$, and $f(x, y) \geq 0$ for all (x, y) in R, then we may interpret

$$\int\int_R f(x, y)dA$$

as the volume of the region

$$S = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, 0 \leq z \leq f(x, y)\}.$$



Graph of $y = 16 - x^2 - y^2$ over $[0, 1] \times [0, 2]$

Example Suppose $f(x, y) = 16 - x^2 - y^2$ and $R = [0, 1] \times [0, 2]$. If we divide $[0, 1]$ into 2 intervals and $[0, 2]$ into 4 intervals and evaluate $f(x, y)$ at the lower left-hand corner of each of the 8 resulting rectangles, then we have the approximation

$$\begin{aligned}\int\int_R f(x, y)dA &\approx f(0, 0) \times \frac{1}{4} + f\left(\frac{1}{2}, 0\right) \times \frac{1}{4} + f\left(0, \frac{1}{2}\right) \times \frac{1}{4} + f\left(\frac{1}{2}, \frac{1}{2}\right) \times \frac{1}{4} \\ &\quad + f(0, 1) \times \frac{1}{4} + f\left(\frac{1}{2}, 1\right) \times \frac{1}{4} + f\left(0, \frac{3}{2}\right) \times \frac{1}{4} + f\left(\frac{1}{2}, \frac{3}{2}\right) \times \frac{1}{4} \\ &= 4 + \frac{63}{16} + \frac{63}{16} + \frac{31}{8} + \frac{15}{4} + \frac{59}{16} + \frac{55}{16} + \frac{54}{16} \\ &= 30.\end{aligned}$$

We shall see that, exactly,

$$\int\int_R (16 - x^2 - y^2)dA = \frac{86}{3},$$

the volume of the region under the graph of f and above the rectangle R in the xy-plane.