Lecture 17: Constrained Extrema

17.1 Lagrange multipliers

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are differentiable and we are looking for the extreme values of f restricted to the level set $S = \{\mathbf{x} : g(\mathbf{x}) = 0\}$. Note that if f has an extreme value at $\mathbf{a}$ on S, then $f(\mathbf{a})$ must be an extreme value of f along any curve passing through $\mathbf{a}$. Hence if $\varphi : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve which lies in S with $\varphi(t_0) = \mathbf{a}$, then the function $h(t) = f(\varphi(t))$ has a local extremum at t_0. Thus

$$0 = h'(t_0) = \nabla f(\varphi(t_0)) \cdot \varphi'(t_0) = \nabla f(\mathbf{a}) \cdot \varphi'(t_0).$$

Now since $\varphi'(t_0)$ is tangent to S and $\nabla f(\mathbf{a})$ is orthogonal to $\varphi'(t_0)$ for any such curve, it follows that $\nabla f(\mathbf{a})$ is orthogonal to S. But we already know that $\nabla g(\mathbf{a})$ is orthogonal to S, and so $\nabla f(\mathbf{a})$ and $\nabla g(\mathbf{a})$ must be parallel. That is, there must exist a scalar λ such that

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a}).$$

Hence to find the extreme values of f restricted to S, we need consider only those points $\mathbf{a}$ for which both

$$g(\mathbf{a}) = 0$$

and

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a}).$$

We call the scalar λ a *Lagrange multiplier* and this method for finding extreme values of a function f subject to a constraining equation $g(\mathbf{x}) = 0$ the *method of Lagrange multipliers*.

Example We look for the extreme values of

$$f(x, y) = x^2 + y^2 - x - y + 1$$

on the set

$$S = \{(x, y) : x^2 + y^2 = 1\}.$$

Let $g(x, y) = x^2 + y^2 - 1$. Then

$$\nabla f(x, y) = (2x - 1, 2y - 1)$$

and

$$\nabla g(x, y) = (2x, 2y).$$

Thus we need to solve the equations

$$(2x - 1, 2y - 1) = \lambda(2x, 2y)$$

and

$$x^2 + y^2 = 1.$$

That is, we need to solve

$$\begin{aligned} 2x - 1 &= 2\lambda x \\ 2y - 1 &= 2\lambda y \\ x^2 + y^2 &= 1. \end{aligned}$$

Note first that $\lambda \neq 1$. Then from the first equation we have

$$x = \frac{1}{2(1-\lambda)}$$

and from the second

$$y = \frac{1}{2(1-\lambda)}.$$

Hence $x = y$. From the last equation, it now follows that

$$2x^2 = 1,$$

that is,

$$x = -\frac{1}{\sqrt{2}} \text{ or } x = \frac{1}{\sqrt{2}}.$$

Thus we have two points to consider for extreme values: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Since S is closed and bounded, we know from the Extreme Value Theorem that one of these values is an absolute maximum of f on S and the other an absolute minimum of f on S. Now

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$$

and

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2},$$

so f has an absolute maximum value of $2 + \sqrt{2}$ at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and an absolute minimum value of $2 - \sqrt{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Example Suppose a farmer wishes to construct a rectangular storage bin, without a top, which will hold the largest volume using 300 square meters of material. If we let x and y be the dimensions of the base and z the height of the bin, then we want to maximize the volume

$$V(x, y, z) = xyz$$

on the region $D = \{(x, y, z) : x > 0, y > 0, z > 0\}$ subject to the constraint

$$xy + 2xz + 2yz = 300.$$

If we let $g(x, y, z) = xy + 2xz + 2yz - 300$, then we our problem is to maximize V subject to $g(x, y, z) = 0$. Now

$$\nabla V(x, y, z) = (yz, xz, xy)$$

and

$$\nabla g(x, y, z) = (y + 2z, x + 2z, 2x + 2y),$$

so we need to solve the equations

$$\begin{aligned} yz &= \lambda(y + 2z) \\ xz &= \lambda(x + 2z) \\ xy &= \lambda(2x + 2y) \\ xy + 2xz + 2yz &= 300. \end{aligned}$$

Now the first two equations imply that

$$\lambda = \frac{yz}{y + 2z}$$

and

$$\lambda = \frac{xz}{x + 2z},$$

and so

$$\frac{yz}{y + 2z} = \frac{xz}{x + 2z},$$

from which it follows that

$$xyz + 2yz^2 = xyz + 2xz^2.$$

Hence $2yz^2 = 2xz^2$, and so $x = y$. Substituting this into the third equation yields $x^2 = 4\lambda x$, or $x = 4\lambda$. It follows that $y = 4\lambda$ and, from the first equation,

$$4\lambda z = 4\lambda^2 + 2\lambda z,$$

from which we obtain $z = 2\lambda$. Putting these results into the final equation gives us

$$16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 300,$$

and so

$$\lambda = \sqrt{\frac{300}{48}} = \sqrt{\frac{25}{4}} = \pm\frac{5}{2}.$$

Since $x > 0$, $y > 0$, and $z > 0$, we now have $x = 10$, $y = 10$, and $z = 5$. Unfortunately, in this case our constraining surface is not bounded, nor do we even have a test to determine if we have found the location of a local extreme value. However, our geometric intuition tells us that there should be a maximum, and we conclude that a bin with dimensions 10 meters by 10 meters by 5 meters will maximize volume for the given surface area of 300 square meters.

17.2 Two constraints

Now suppose $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}$ are all differentiable and we wish to find the extreme values of f subject to the constraints $g(\mathbf{x}) = 0$ and $h(\mathbf{x}) = 0$. It follows, as above, that if f has an extremum at $\mathbf{a}$, then $\nabla f(\mathbf{a})$ is orthogonal to any vector tangent to the intersection of the level sets determined by the conditions $g(\mathbf{x}) = 0$ and $h(\mathbf{x}) = 0$. The set of vectors orthogonal to the intersection of these two level sets forms a plane, so it follows that $\nabla f(\mathbf{a})$ lies in the plane spanned by the two vectors $\nabla g(\mathbf{a})$ and $\nabla h(\mathbf{a})$ (provided neither of these vectors is $\mathbf{0}$ and they are not parallel). Hence there must exist scalars λ and μ, again called Lagrange multipliers, such that

$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a}) + \mu \nabla h(\mathbf{a}).$$

Thus to look for possible extreme values of f, we solve the equations

$$\begin{aligned} \nabla f(\mathbf{x}) &= \lambda \nabla g(\mathbf{x}) + \mu \nabla h(\mathbf{x}) \\ g(\mathbf{x}) &= 0 \\ h(\mathbf{x}) &= 0. \end{aligned}$$

Example Suppose the unit sphere centered at the origin in $\mathbb{R}^3$ is heated so that its temperature at a point is given by

$$T(x, y, z) = 80 + 50(x + z).$$

Suppose we wish to find the extreme values of T along the intersection D of the sphere with the plane $x + y + z = 1$ (see the picture below). That is, we wish to find the extrema of T subject to the constraints

$$x^2 + y^2 + z^2 = 1$$

and

$$x + y + z = 1.$$

Let $g(x, y, z) = x^2 + y^2 + z^2 - 1$ and $h(x, y, z) = x + y + z - 1$. Now

$$\nabla T(x, y, z) = (50, 0, 50),$$

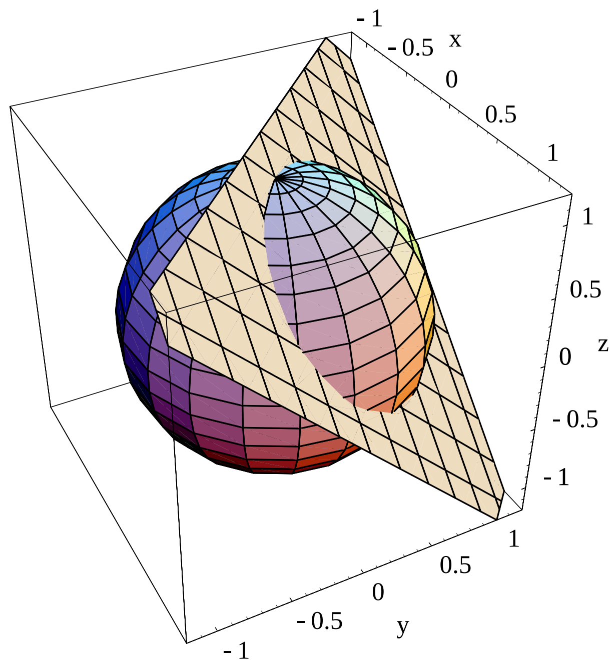
$$\nabla g(x, y, z) = (2x, 2y, 2z),$$

and

$$\nabla h(x, y, z) = (1, 1, 1).$$

So we need to solve the equations

$$(50, 0, 50) = \lambda(2x, 2y, 2z) + \mu(1, 1, 1),$$



Intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$

$$x^2 + y^2 + z^2 = 1$$

and

$$x + y + z = 1.$$

That is, we need to solve

$$\begin{aligned} 50 &= 2\lambda x + \mu \\ 0 &= 2\lambda y + \mu \\ 50 &= 2\lambda z + \mu \\ x^2 + y^2 + z^2 &= 1 \\ x + y + z &= 1. \end{aligned}$$

Note first that we cannot have $\lambda = 0$. From the first and third equations we obtain

$$2\lambda x + \mu = 2\lambda z + \mu,$$

from which it follows that $x = z$. It now follows from the last equation that

$$y = 1 - 2x.$$

Substituting into the fourth equation, we have

$$x^2 + (1 - 2x)^2 + x^2 = 1,$$

that is,

$$6x^2 - 4x = 0.$$

Hence $x = 0$ or $x = \frac{2}{3}$. When $x = 0$, $z = 0$ and $y = 1$; when $x = \frac{2}{3}$, $z = \frac{2}{3}$ and $y = -\frac{1}{3}$. Since D is closed and bounded, we evaluate T at these two points, namely,

$$T(0, 1, 0) = 80$$

and

$$T\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = 80 + \frac{200}{3} = \frac{440}{3},$$

and see that T has an absolute minimum value of 80 at $(0, 1, 0)$ and an absolute maximum value of $\frac{440}{3}$ at $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$.