Lecture 16: Extreme Values

16.1 Open and closed sets

Definition Given a point $\mathbf{a}$ in $\mathbb{R}^n$ and a scalar $r > 0$, we call the set

$$B(\mathbf{a}, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| < r\}$$

the *open ball* of radius r centered at $\mathbf{a}$.

Definition We say a set U in $\mathbb{R}^n$ is *open* if for every point $\mathbf{a}$ in U there exists a scalar $r > 0$ such that the open ball $B(\mathbf{a}, r)$ is contained in U.

Definition Given a set D in $\mathbb{R}^n$, if the open ball $B(\mathbf{a}, r)$ contains both points in D and points not in D for all $r > 0$, then we call $\mathbf{a}$ a *boundary point* of D. We call the set of all boundary points of D the *boundary* of D. If the boundary of D is a subset of D, then we say D is *closed.*

Example An open ball $B(\mathbf{a}, r)$ is an open set. The boundary of $B(\mathbf{a}, r)$ is the sphere

$$S(\mathbf{a}, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| = r\}.$$

The *closed ball*

$$C(\mathbf{a}, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{a}| \leq r\}$$

is a closed set, as is the sphere itself.

Example The set

$$D = \{(x, y) : 0 < x < 2, 0 < y < 3\}$$

is open. We call D an *open rectangle.* The boundary of D is the enclosing rectangle, which is a closed set, as it the *closed rectangle*

$$R = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}$$

16.2 Local extrema

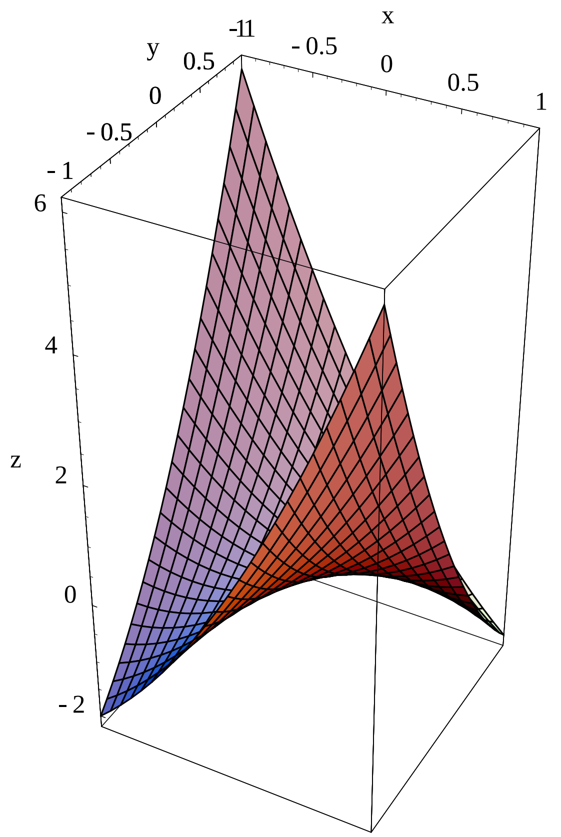
Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$. If $f(\mathbf{a}) \leq f(\mathbf{x})$ for all x in some open ball centered at $\mathbf{a}$, then we say f has a *local minimum* at $\mathbf{a}$. If $f(\mathbf{a}) \geq f(\mathbf{x})$ for all x in some open ball centered at $\mathbf{a}$, then we say f has a *local maximum* at $\mathbf{a}$. If f has either a local maximum or a local minimum at $\mathbf{a}$, then we say f has a *local extremum* at $\mathbf{a}$.

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ and f has a local extremum at $\mathbf{a}$, then $\nabla f(\mathbf{a}) = \mathbf{0}$.

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$. If f is differentiable at $\mathbf{a}$ and $\nabla f(\mathbf{a}) = \mathbf{0}$, then we say $\mathbf{a}$ is a *stationary point* of f. If $\mathbf{a}$ is a point at which f is not differentiable, then we call $\mathbf{a}$ a *singular point* of f. If $\mathbf{a}$ is either a stationary point or a singular point of f, then we call $\mathbf{a}$ a *critical point.* A stationary point at which f does not have a local extremum is called a *saddle point* of f.

Recall that if $f : \mathbb{R} \to \mathbb{R}$, $f'(a) = 0$, and $f''(a) > 0$, then f has a local minimum it a (a local maximum if $f''(a) < 0$). Now suppose $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{a}$ is a stationary point of f, and both $f_{xx}(\mathbf{a}) > 0$ and $f_{yy}(\mathbf{a}) > 0$. Then f has a local minimum at $\mathbf{a}$ in directions parallel to the x and y axes. However, this is not sufficient to guarantee that f has a local minimum at $\mathbf{a}$, as the next example demonstrates.

Example If $f(x, y) = x^2 + y^2 - 4xy$, then $\nabla f(x, y) = (2x - 4y, 2y - 4x)$, $\nabla f(0, 0) = (0, 0)$, $f_{xx}(0, 0) = 2 > 0$, and $f_{yy}(0, 0) = 2 > 0$. Hence f has a local minimum along both the x- and y-axes at $(0, 0)$. However, if we let $g(t) = f(t, t)$ (that is, g takes the values of f along the line $x = y$), then $g(t) = -2t^2$, which has a local maximum at $t = 0$. Hence f has a local maximum at $(0, 0)$ along the line $x = y$. Thus f has a saddle point at $(0, 0)$.



Graph of $f(x, y) = x^2 + y^2 - 4xy$

Second Derivative Test Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has a stationary point at (a, b) and the second partial derivatives of f are all continuous on an open disk centered at (a, b). Let

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

Then f has a local minimum at (a, b) if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, a local maximum at (a, b) if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, and (a, b) is a saddle point of f if $D(a, b) < 0$.

Note that $D(a, b)$ is the determinant of the matrix

$$\begin{bmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{bmatrix},$$

which we call the *Hessian* of f.

Example If $f(x, y) = x^2 + y^2 - 4xy$, then

$$f_{xy}(x, y) = -4,$$

so

$$D(0, 0) = (2)(2) - 16 = -12 < 0,$$

showing once again that $(0, 0)$ is a saddle point of f.

Example Consider

$$f(x, y) = xye^{-(x^2+y^2)}.$$

Then

$$f_x(x, y) = -2x^2ye^{-(x^2+y^2)} + xe^{-(x^2+y^2)} = y(1 - 2x^2)e^{-(x^2+y^2)}$$

and

$$f_y(x, y) = -2xy^2e^{-(x^2+y^2)} + ye^{-(x^2+y^2)} = x(1 - 2y^2)e^{-(x^2+y^2)},$$

so $\nabla f(x, y) = (0, 0)$ when

$$y(1 - 2x^2) = 0$$
$$x(1 - 2y^2) = 0.$$

Hence the critical points are $(0, 0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Now

$$f_{xx}(x, y) = (4x^3y - 6xy)e^{-(x^2+y^2)},$$

$$f_{xy}(x, y) = (4x^2y^2 - 2x^2 - 2y^2 + 1)e^{-(x^2+y^2)},$$

and

$$f_{yy}(x, y) = (4xy^3 - 6xy)e^{-(x^2+y^2)},$$

so

$$D(0, 0) = (0)(0) - 1^2 = -1 < 0,$$

$$D\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (-2e^{-1})(-2e^{-1}) - 0^2 = 4e^{-2} > 0,$$

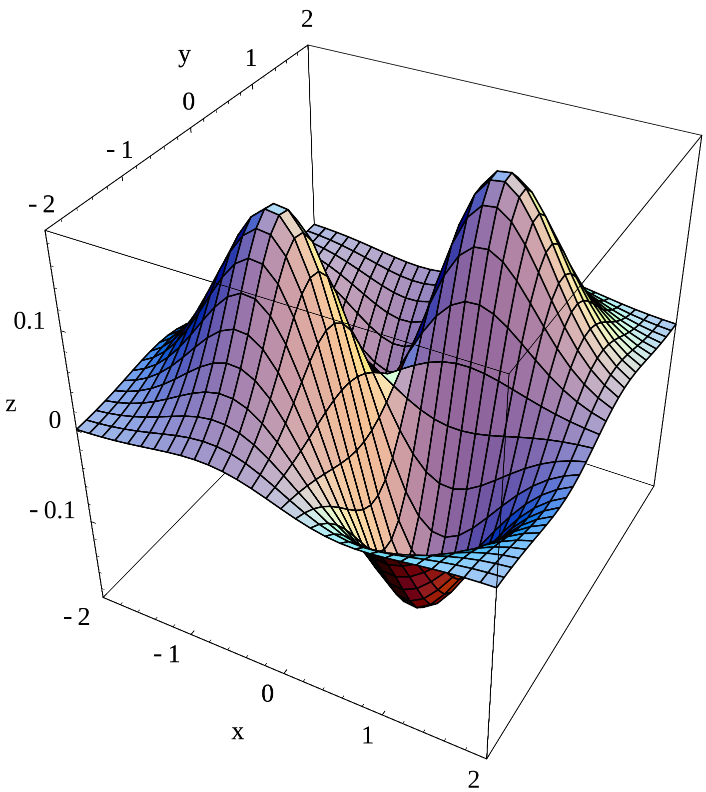
$$D\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (2e^{-1})(2e^{-1}) - 0^2 = 4e^{-2} > 0,$$

$$D\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (2e^{-1})(2e^{-1}) - 0^2 = 4e^{-2} > 0,$$

and

$$D\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (-2e^{-1})(-2e^{-1}) - 0^2 = 4e^{-2} > 0.$$

Thus f has a local maximums of $\frac{1}{2}e^{-1}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, local minimums of $-\frac{1}{2}e^{-1}$ at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and $(0,0)$ is a saddle point.



Graph of $f(x,y) = xye^{-(x^2+y^2)}$

16.3 Absolute extrema

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ has domain S. If $f(\mathbf{a}) \le f(\mathbf{x})$ for all $\mathbf{x}$ in S, then we say f has an *absolute minimum* at $\mathbf{a}$. If $f(\mathbf{a}) \ge f(\mathbf{x})$ for all $\mathbf{x}$ in S, then we say f has an *absolute maximum* at $\mathbf{a}$. If f has either an absolute minimum or an absolute maximum at $\mathbf{a}$, then we say f has an *absolute extremum* at $\mathbf{a}$.

Definition We say a set S in $\mathbb{R}^n$ is *bounded* if S is a subset of the open ball $B(\mathbf{0}, r)$ for some scalar r.

Extreme Value Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on a closed bounded set S, then f attains an absolute maximum at some point $\mathbf{a}$ in S and f attains an absolute minimum at some point $\mathbf{b}$ in S.

Example Consider the function $f(x, y) = x^2 + y^2 - x - y + 1$ defined on the closed disk $S = \{(x, y) : x^2 + y^2 \leq 1\}$. To find the absolute extreme values of f, we first find the critical values of f. Now

$$f_x(x, y) = 2x - 1$$

and

$$f_y(x, y) = 2y - 1,$$

so $\nabla f(x, y) = (0, 0)$ when $(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$. To check the boundary of S, we parametrize it by $\varphi(t) = (\cos(t), \sin(t))$ for $0 \leq t \leq 2\pi$ and let

$$g(t) = f(\varphi(t)) = \cos^2(t) + \sin^2(t) - \cos(t) - \sin(t) + 1 = 2 - \cos(t) - \sin(t).$$

Now $g'(t) = \sin(t) - \cos(t)$, and so $g'(t) = 0$ when $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$. Hence the extreme values of f must occur at $\left(\frac{1}{2}, \frac{1}{2}\right)$,

$$\varphi\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$\varphi\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),$$

or

$$g(0) = g(2\pi) = (1, 0).$$

Now

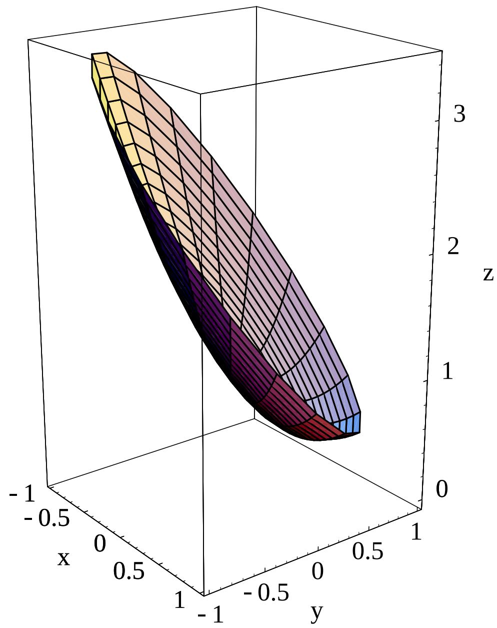
$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2},$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2},$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 2 + \sqrt{2},$$

and

$$f(1, 0) = 1.$$



Graph of $f(x,y) = x^2 + y^2 - x - y + 1$ on $\{(x,y) : x^2 + y^2 \leq 1\}$

Hence f has an absolute minimum value of $\frac{1}{2}$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and an absolute maximum value of $2 + \sqrt{2}$ at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Example A farmer wishes to build a rectangular storage bin, without a top, with a volume of 500 cubic meters using the least amount of material possible. If we let x and y be the dimensions of the base of the bin and z be the height, all measured in meters, then the farmer wishes to minimize the surface area of the bin, given by

$$S = xy + 2xz + 2yz,$$

subject to the constraint on the volume, namely,

$$500 = xyz.$$

Solving for z in the latter expression and substituting into the expression for S, we have

$$S = xy + \frac{1000}{y} + \frac{1000}{x}.$$

Our problem is then to minimize S over the region

$$T = \{(x,y) : x > 0, y > 0\}.$$

Now

$$\frac{\partial S}{\partial x} = y - \frac{1000}{x^2}$$

and

$$\frac{\partial S}{\partial y} = x - \frac{1000}{y^2},$$

so we need to solve the pair of equations

$$\begin{aligned} y - \frac{1000}{x^2} &= 0 \\ x - \frac{1000}{y^2} &= 0. \end{aligned}$$

Solving for y in the first of these, we have

$$y = \frac{1000}{x^2};$$

substituting into the second gives us

$$0 = x - \frac{x^4}{1000} = x\left(1 - \frac{x^3}{1000}\right),$$

which has solutions $x = 0$ and $x = 10$. The first of these will not give solutions in T, and from the second we obtain

$$y = \frac{1000}{10^2} = 10.$$

Hence we have the single stationary point $(10, 10)$. Now

$$\frac{\partial^2 S}{\partial x^2} = \frac{2000}{x^3},$$

$$\frac{\partial^2 S}{\partial y \partial x} = 1,$$

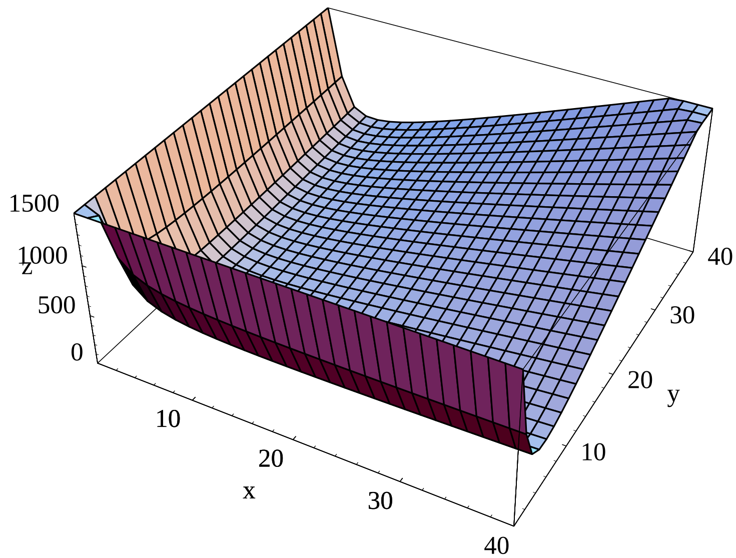
and

$$\frac{\partial^2 S}{\partial y^2} = \frac{2000}{y^3},$$

so

$$D(10, 10) = (2)(2) - 1^2 = 3.$$

Hence S has a local minimum at $(10, 10)$. Although this does not guarantee that S has an absolute minimum at $(10, 10)$ (as it would in the analogous one-dimensional case), nevertheless it is indicated from the fact that S grows without bound as (x, y) approaches either axis or as $|(x, y)|$ increases (see the graph below). Hence we conclude that the dimensions $x = 10$ meters, $y = 10$ meters, and $z = 5$ meters will minimize the amount of material required to construct the bin.



Graph of $S = xy + \dfrac{1000}{y} + \dfrac{1000}{x}$