Lecture 15: The Gradient

15.1 Directional derivatives revisited

Recall: If $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{u}$ is a unit vector, then the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h},$$

provided the limit exists. Now if we define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = f(\mathbf{a} + t\mathbf{u}),$$

then

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h} = D_{\mathbf{u}} f(\mathbf{a}).$$

Moreover, by the chain rule, we have

$$g'(0) = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Hence we have the following theorem.

Theorem If f is differentiable at $\mathbf{a}$ and $\mathbf{u}$ is unit vector, then

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Example Let $f(x, y) = 4 - x^2 - y^2$. Then

$$\nabla f(x, y) = (-2x, -2y).$$

If

$$\mathbf{u} = \frac{1}{\sqrt{2}}(-1, -1),$$

then

$$D_{\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot \mathbf{u} = (-2, -2) \cdot \frac{1}{\sqrt{2}}(-1, -1) = \frac{4}{\sqrt{2}} = 2\sqrt{2},$$

a result we found earlier by a direct computation. Note that

$$D_{-\mathbf{u}} f(1, 1) = \nabla f(1, 1) \cdot (-\mathbf{u}) = -(\nabla f(1, 1) \cdot \mathbf{u}) = -D_{\mathbf{u}} f(1, 1) = -2\sqrt{2}.$$

Note that the final calculation in the example holds in general:

$$D_{-\mathbf{u}} f(\mathbf{a}) = -D_{\mathbf{u}} f(\mathbf{a}).$$

15.2 Direction of maximum increase

If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ and $\mathbf{u}$ is a unit vector, then

$$|D_{\mathbf{u}} f(\mathbf{a})| = |\nabla f(\mathbf{a}) \cdot \mathbf{u}| \leq |\nabla f(\mathbf{a})||\mathbf{u}| = |\nabla f(\mathbf{a})|,$$

with equality if and only if $\mathbf{u}$ and $\nabla f(\mathbf{a})$ are parallel. Hence $D_{\mathbf{u}} f(\mathbf{a})$ attains a maximum value of $|\nabla f(\mathbf{a})|$ when

$$\mathbf{u} = \frac{1}{|\nabla f(\mathbf{a})|} \nabla f(\mathbf{a}),$$

and $D_{\mathbf{u}} f(\mathbf{a})$ attains a minimum value of $-|\nabla f(\mathbf{a})|$ when

$$\mathbf{u} = -\frac{1}{|\nabla f(\mathbf{a})|} \nabla f(\mathbf{a}).$$

Example A metal plate is heated so that its temperature at a point (x, y) is given by

$$T(x, y) = 100x^2 e^{-\frac{1}{20}(x^2+y^2)}.$$

Then

$$\frac{\partial}{\partial x} T(x, y) = -10x^3 e^{-\frac{1}{20}(x^2+y^2)} + 200x e^{-\frac{1}{20}(x^2+y^2)} = 10x e^{-\frac{1}{20}(x^2+y^2)}(20 - x^2)$$

and

$$\frac{\partial}{\partial y} T(x, y) = -10x^2 y e^{-\frac{1}{20}(x^2+y^2)}.$$

Thus

$$\nabla T(x, y) = 10x e^{-\frac{1}{20}(x^2+y^2)}(20 - x^2, -xy).$$

Hence, for example,

$$\nabla T(1, 2) = 10e^{-\frac{1}{4}}(19, -2).$$

If we let

$$\mathbf{u} = \frac{1}{\sqrt{365}}(19, -2),$$

then, from the point $(1, 2)$, the temperature is increasing most rapidly in the direction of $\mathbf{u}$ and is decreasing most rapidly in the direction of $-\mathbf{u}$. Moreover, the rate of increase in the direction of $\mathbf{u}$ is

$$|\nabla T(1, 2)| = 10\sqrt{365}e^{-\frac{1}{4}} \approx 148.79;$$

the rate of increase in the direction of $-\mathbf{u}$ is

$$-|\nabla T(1, 2)| = -10\sqrt{365}e^{-\frac{1}{4}} \approx -148.79.$$

15.3 The gradient and level sets

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$, let $c = f(\mathbf{a})$, and let L be the level set of all points in $\mathbb{R}^n$ satisfying $f(\mathbf{x}) = c$. If $\varphi : \mathbb{R} \to \mathbb{R}^n$ parametrizes a curve C which lies entirely in L with $\varphi(t_0) = \mathbf{a}$, then $f(\varphi(t)) = c$ for all t. Hence

$$0 = \frac{d}{dt} f(\varphi(t)) = \nabla f(\varphi(t)) \cdot \varphi'(t)$$

for all t. In particular,

$$\nabla f(\mathbf{a}) \cdot \varphi'(t_0) = 0.$$

Since φ was arbitrary, this says that $\nabla f(\mathbf{a})$ is orthogonal to any vector which is tangent to L at $\mathbf{a}$. In particular, for $n = 2$, $\nabla f(\mathbf{a})$ is orthogonal to the line tangent to the level curve through $\mathbf{a}$, and, for $n = 3$, $\nabla f(\mathbf{a})$is orthogonal to the plane tangent to the level surface through $\mathbf{a}$.

Example Let E be the ellipse with equation

$$x^2 + 4y^2 = 16.$$

Then E is a level curve of the function

$$f(x, y) = x^2 + 4y^2.$$

To find an equation of the line tangent to E at $(2, \sqrt{3})$, we find that

$$\nabla f(x, y) = (2x, 8y)$$

and

$$\nabla f(2, \sqrt{3}) = (4, 8\sqrt{3}).$$

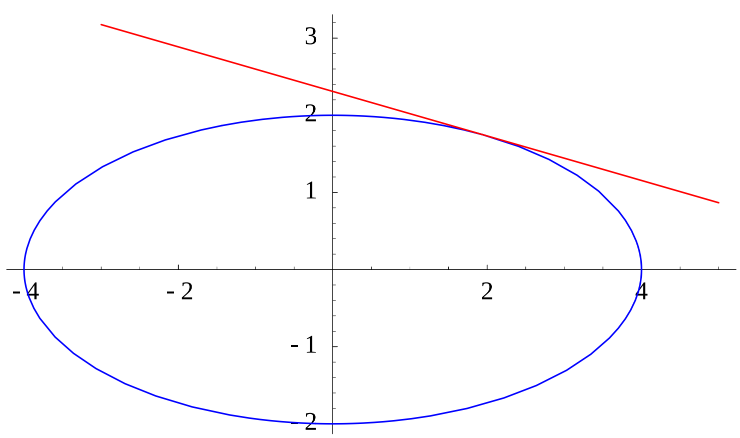
Since $\nabla f(2, \sqrt{3})$ is orthogonal to the tangent line at $(2, \sqrt{3})$, an equation for the tangent line is given by

$$(4, 8\sqrt{3}) \cdot (x - 2, y - \sqrt{3}) = 0.$$

Expanding, we see that

$$4x + 8\sqrt{3}y = 32$$

is an equation for the line tangent to E at $(2, \sqrt{3})$.



Ellipse with tangent line

Example Let E be the ellipsoid with equation

$$x^2 + 2y^2 + 4z^2 = 21.$$

Then E is a level surface of the function

$$f(x, y) = x^2 + 2y^2 + 4z^2.$$

To find an equation of the plane tangent to E at $(3, 2, 1)$, we find that

$$\nabla f(x, y, z) = (2x, 4y, 8z)$$

and

$$\nabla f(3, 2, 1) = (6, 8, 8).$$

Since $\nabla f(3, 2, 1)$ is orthogonal to the tangent plane at $(3, 2, 1)$, an equation for the tangent plane is given by

$$(6, 8, 8) \cdot (x - 3, y - 2, z - 1) = 0.$$

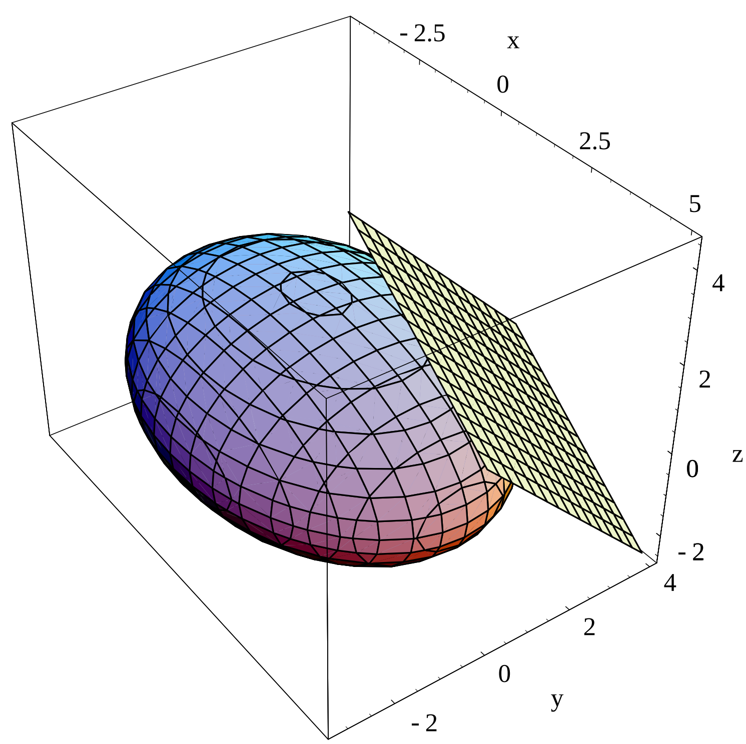
Expanding, we see that

$$6x + 8y + 8z = 42,$$

or, equivalently,

$$3x + 4y + 4z = 21,$$

is an equation for the plane tangent to E at $(3, 2, 1)$.



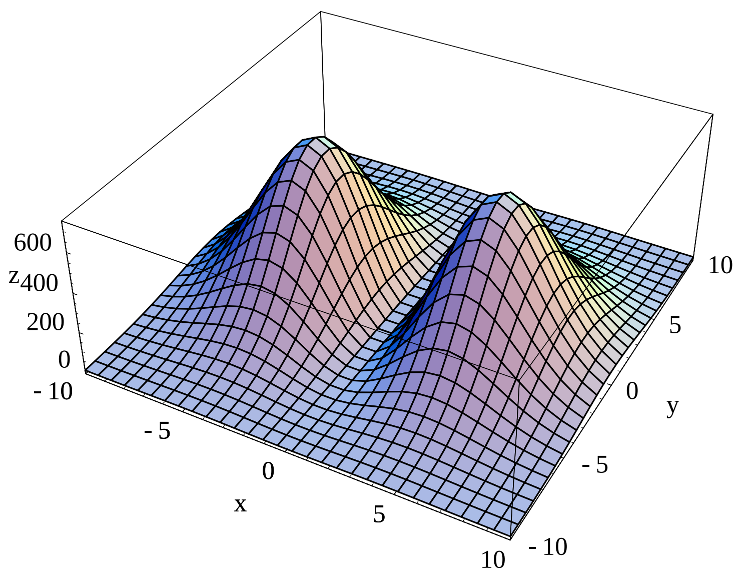
Ellipsoid with tangent plane

15.4 Plotting vector fields

Because of the properties of the gradient vector discussed above, a plot of the gradient vectors on a lattice of points can be very helpful in visualizing a function.

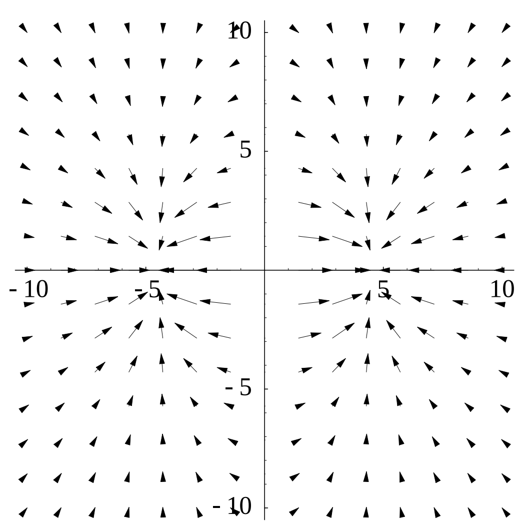
Example Here is a plot of the surface

$$T(x,y) = 100x^2 e^{-\frac{1}{20}(x^2+y^2)}:$$



Graph of $T(x,y) = 100x^2 e^{-\frac{1}{20}(x^2+y^2)}$

and here is the corresponding plot of gradient vectors:



Gradient vector field of $T(x,y) = 100x^2 e^{-\frac{1}{20}(x^2+y^2)}$