Lecture 13: Derivatives

13.1 Partial derivatives do not imply differentiability

Recall that for functions $f : \mathbb{R} \to \mathbb{R}$, differentiability is a stronger condition than continuity. That is, if f is differentiable at a, then f is continuous at a. Geometrically, you cannot hope to have a tangent line at a point on the graph of f if the curve is not even continuous. Similarly, we would not want to say a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point if the graph of f has a tear, and hence does not have a tangent plane, at the given point. Now if

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0) \end{cases},$$

then f is not continuous at $(0, 0)$ (we showed earlier that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist), even though both $f_x(0, 0)$ and $f_y(0, 0)$ exist:

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Hence for functions $f : \mathbb{R}^n \to \mathbb{R}$, the existence of partial derivatives is not enough to ensure that a function is differentiable.

13.2 The idea of a derivative

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a. We often think of

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

as the slope of the graph of f at a. This idea of a derivative does not generalize well to higher dimensions since, as we have seen, the graph of a function of several variables may have different slopes depending on the direction chosen. On the other hand, consider the linear function $L(x) = f'(a)x$. Then L has the property that

$$\lim_{h\to 0} \frac{f(a + h) - f(a) - L(h)}{h} = \lim_{h\to 0} \left(\frac{f(a + h) - f(a)}{h} - f'(a) \right) = f'(a) - f'(a) = 0.$$

Definition We call a function $L : \mathbb{R}^n \to \mathbb{R}$ *linear* if there exist scalars $a_1, a_2, \ldots, a_n$ such that

$$L(x_1, x_2, \ldots, x_n) = a_1x_n + a_2x_2 + \ldots + a_nx_n$$

for all $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$.

Definition We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is *differentiable* at a point $\mathbf{a}$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})}{|h|} = 0,$$

in which case we call L the *derivative* or *differential* of f, often denoted $df_{\mathbf{a}}$.

Now if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point $\mathbf{a} = (a_1, a_2)$, then

$$\begin{aligned} \mathbf{n}(a_1, a_2) &= \left(1, 0, \frac{\partial}{\partial x} f(a_1, a_2)\right) \times \left(0, 1, \frac{\partial}{\partial y} f(a_1, a_2)\right) \\ &= \left(-\frac{\partial}{\partial x} f(a_1, a_2), -\frac{\partial}{\partial y} f(a_1, a_2), 1\right). \end{aligned}$$

should be a normal vector for the plane tangent to the graph of f at $(a_1, a_2, f(a_1, a_2))$. That is, the equation of the tangent plane should be

$$\mathbf{n}(a_1, a_2) \cdot (x - a_1, y - a_2, z - f(a_1, a_2)) = 0,$$

which simplifies to

$$z = f(a_1, a_2) + \frac{\partial}{\partial x} f(a_1, a_2)(x - a_1) + \frac{\partial}{\partial x} f(a_1, a_2)(y - a_2).$$

This motivates a guess as to what the derivative of a function $f : \mathbb{R}^2 \to \mathbb{R}$ should be if one exists.

Theorem If $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives on an open disk centered at $\mathbf{a} = (a_1, a_2)$, then f is differentiable at $\mathbf{a}$. Moreover,

$$df_{\mathbf{a}}(x, y) = \frac{\partial f}{\partial x}(a_1, a_2)x + \frac{\partial f}{\partial y}(a_1, a_2)y.$$

Proof Let $\mathbf{h} = (h_1, h_2)$ and define $L : \mathbb{R}^2 \to \mathbb{R}$ by

$$L(x, y) = \frac{\partial f}{\partial x}(a_1, a_2)x + \frac{\partial f}{\partial y}(a_1, a_2)y.$$

Now

$$\begin{aligned} f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) &= f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) \\ &= (f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2)) + (f(a_1 + h_1, a_2) - f(a_1, a_2)). \end{aligned}$$

Using the mean value theorem from single-variable calculus, we know that there is a c_1 between 0 and h_1 and a c_2 between 0 and h_2 such that

$$f(a_1 + h_1, a_2) - f(a_1, a_2) = \frac{\partial}{\partial x} f(a_1 + c_1, a_2) h_1$$

and

$$f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2) = \frac{\partial}{\partial y} f(a_1 + h_1, a_2 + c_2) h_2.$$

Hence

$$\begin{aligned} f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) - L(h_1, h_2) &= \left(\frac{\partial}{\partial y} f(a_1 + h_1, a_2 + c_2) - \frac{\partial}{\partial y} f(a_1, a_2) \right) h_2 \\ &\quad + \left(\frac{\partial}{\partial x} f(a_1 + c_1, a_2) - \frac{\partial}{\partial x} f(a_1, a_2) \right) h_1, \end{aligned}$$

and so

$$\begin{aligned} &|f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) - L(h_1, h_2)| \\ &\quad \leq \left| \left(\frac{\partial}{\partial x} f(a_1 + c_1, a_2) - \frac{\partial}{\partial x} f(a_1, a_2), \frac{\partial}{\partial y} f(a_1 + h_1, a_2 + c_2) - \frac{\partial}{\partial y} f(a_1, a_2) \right) \right| |(h_1, h_2)|. \end{aligned}$$

Hence

$$\begin{aligned} &\left| \frac{f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) - L(h_1, h_2)}{|(h_1, h_2)|} \right| \\ &\quad \leq \left| \left(\frac{\partial}{\partial x} f(a_1 + c_1, a_2) - \frac{\partial}{\partial x} f(a_1, a_2), \frac{\partial}{\partial y} f(a_1 + h_1, a_2 + c_2) - \frac{\partial}{\partial y} f(a_1, a_2) \right) \right|. \end{aligned}$$

Since the partial derivatives are continuous, we have

$$\lim_{\mathbf{h} \to \mathbf{0}} \left(\frac{\partial}{\partial x} f(a_1 + c_1, a_2) - \frac{\partial}{\partial x} f(a_1, a_2) \right) = 0$$

and

$$\lim_{\mathbf{h} \to \mathbf{0}} \left(\frac{\partial}{\partial y} f(a_1 + h_1, a_2 + c_2) - \frac{\partial}{\partial y} f(a_1, a_2) \right) = 0.$$

Hence

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})}{|\mathbf{h}|} = 0,$$

and so f is differentiable with derivative L.

In general, we have the following result.

Theorem If $f : \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives on an open ball centered at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$. Moreover,

$$df_{\mathbf{a}}(x_1, x_2, \ldots, x_n) = \frac{\partial f}{\partial x_1}(\mathbf{a})x_1 + \frac{\partial f}{\partial x_2}(\mathbf{a})x_2 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{a})x_n.$$

13.3 Linear approximations

Definition If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, then we call the function $L : \mathbb{R}^n \to \mathbb{R}$ defined by

$$L(x_1, x_2, \ldots, x_n) = f(\mathbf{a}) + \frac{\partial f}{\partial x_1}(\mathbf{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\mathbf{a})(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{a})(x_n - a_n)$$

the *linearization* of f at $\mathbf{a}$.

Example If $f(x, y, z) = \ln(x^2 + y^2 + z^2)$, then

$$\frac{\partial}{\partial x} f(x, y, z) = \frac{2x}{x^2 + y^2 + z^2},$$

$$\frac{\partial}{\partial y} f(x, y, z) = \frac{2y}{x^2 + y^2 + z^2},$$

and

$$\frac{\partial}{\partial z} f(x, y, z) = \frac{2z}{x^2 + y^2 + z^2}.$$

Hence

$$\frac{\partial}{\partial x} f(1, 2, -2) = \frac{2}{9},$$

$$\frac{\partial}{\partial y} f(1, 2, -2) = \frac{4}{9},$$

and

$$\frac{\partial}{\partial z} f(1, 2, -2) = -\frac{4}{9},$$

so the linearization of f at $(1, 2, -2)$ is

$$L(x, y, z) = \ln(9) + \frac{2}{9}(x - 1) + \frac{4}{9}(y - 2) - \frac{4}{9}(z + 2).$$

For example, we might estimate

$$f(1.1, 1.9, -2.1) \approx L(1.1, 1.9, -2.1) = \ln(3) + \frac{2}{90} - \frac{4}{90} + \frac{4}{90} = \ln(3) + \frac{2}{90}.$$

13.4 Tangent planes

Definition If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then we call the graph of the linearization of f at (a, b) the *tangent plane* to the graph of f at $(a, b, f(a, b))$.

Example If $f(x, y) = 9 - x^2 - y^2$, then

$$\frac{\partial}{\partial x} f(x, y) = -2x$$

and

$$\frac{\partial}{\partial y} f(x, y) = -2y,$$

so

$$\frac{\partial}{\partial x} f(1, 2) = -2$$

and

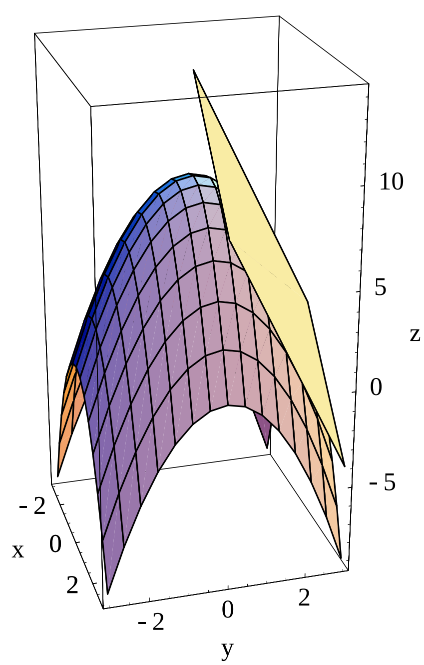
$$\frac{\partial}{\partial x} f(1, 2) = -4.$$

So the linearization of f at $(1, 2)$ is

$$L(x, y) = 4 - 2(x - 1) - 4(y - 2) = 14 - 2x - 4y.$$

Hence the tangent plane to the graph of f at $(1, 2, 4)$ has equation

$$z = 14 - 2x - 4y.$$



Graph of $f(x, y) = 9 - x^2 - y^2$ with tangent plane at $(1, 2, 4)$