Lecture 12: Directional and Partial Derivatives

12.1 Directional derivatives

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x}$ is a point in the domain of f, and $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$. Providing the limit exists, we call

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}$$

the *directional derivative* of f in the direction of $\mathbf{u}$.

Example If $f(x, y) = 4 - x^2 - y^2$ and

$$\mathbf{u} = \frac{1}{\sqrt{2}}(-1, -1),$$

then

$$\begin{aligned}
D_{\mathbf{u}} f(1,1) &= \lim_{h \to 0} \frac{f((1,1) + h\mathbf{u}) - f(1,1)}{h} \\
&= \lim_{h \to 0} \frac{1}{h}\left(4 - \left(1 - \frac{h}{\sqrt{2}}\right)^2 - \left(1 - \frac{h}{\sqrt{2}}\right)^2 - 2\right) \\
&= \lim_{h \to 0} \frac{1}{h}\left(4 - 2\left(1 - \sqrt{2}h + \frac{h^2}{2}\right) - 2\right) \\
&= \lim_{h \to 0} (2\sqrt{2} - h) \\
&= 2\sqrt{2}.
\end{aligned}$$

The figure below shows the graph of f along with the curve above the point $(1, 1)$ in the xy-plane in the direction of $\mathbf{u}$.

12.2 Partial derivatives

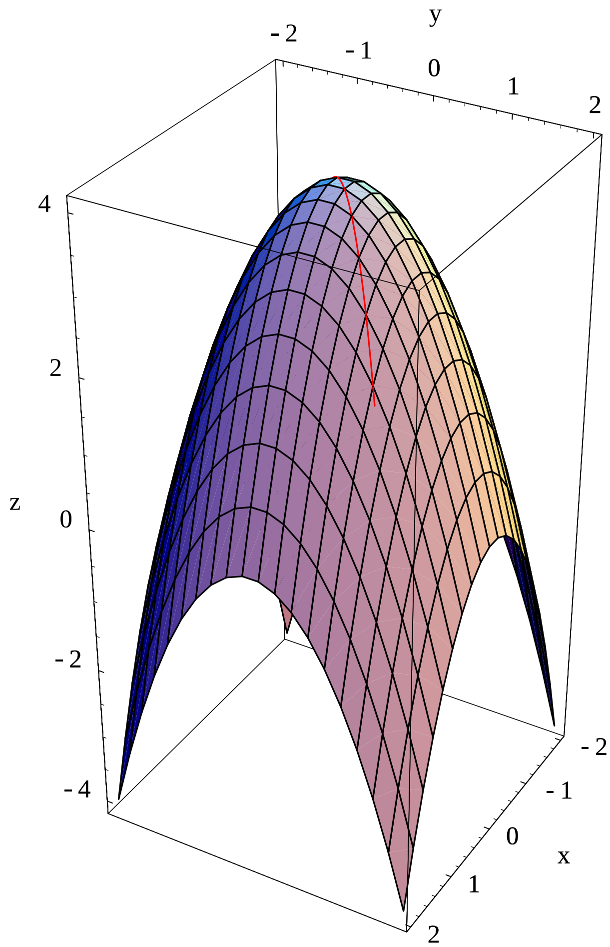
Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and let $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$ be the standard basis vectors for $\mathbb{R}^n$. If $D_{\mathbf{e}_k} f(x_1, x_2, \ldots, x_n)$ exists, then we call $D_{\mathbf{e}_k} f(x_1, x_2, \ldots, x_n)$ the *partial derivative* of f with respect to x_k.

Notations for $D_{\mathbf{e}_k} f(x_1, x_2, \ldots, x_n)$ include:

$$D_{x_k} f(x_1, x_2, \ldots, x_n),$$

$$D_k f(x_1, x_2, \ldots, x_n),$$

$$f_{x_k}(x_1, x_2, \ldots, x_n),$$



Graph of $f(x,y) = 4 - x^2 - y^2$

$$f_k(x_1, x_2, \ldots, x_n),$$

and

$$\frac{\partial}{\partial x_k} f(x_1, x_2, \ldots, x_n).$$

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. Then

$$f_x(x,y) = \lim_{h \to 0} \frac{f((x,y) + h(1,0)) - f(x,y)}{h} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

If, fixing y, we let $g(x) = f(x,y)$, then

$$f_x(x,y) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x).$$

That is, we may compute f_x by treating y as fixed and differentiating with respect to x. A similar comment holds for all partial derivatives: to find the partial derivative of

$f : \mathbb{R}^n \to \mathbb{R}$ with respect to x_k, treat the other variables as fixed and differentiate with respect to x_k as you would a function of one variable.

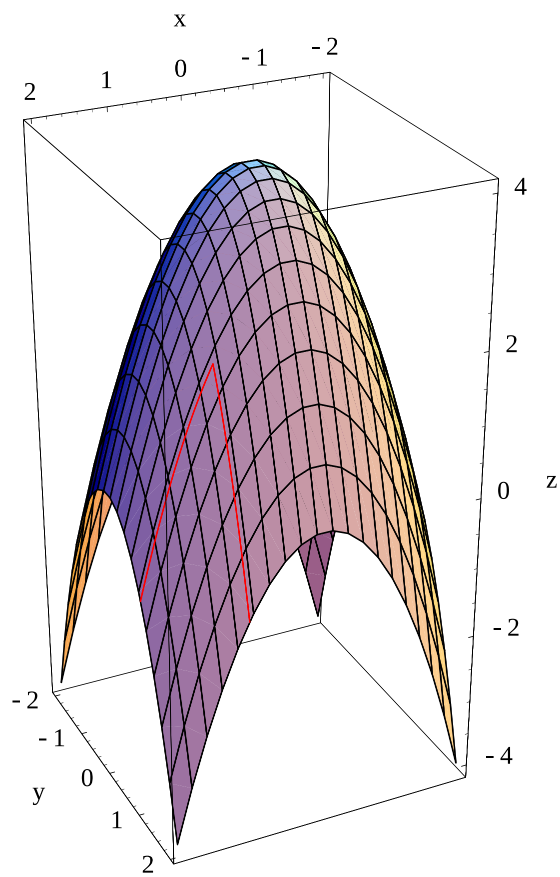
Example If $f(x, y) = 4 - x^2 - y^2$, then

$$f_x(x, y) = -2x$$

and

$$f_y(x, y) = -2y.$$

Hence, for example, $f_x(1, 1) = -2$ and $f_y(x, y) = -2$. The figure below illustrates these slopes.



Graph of $f(x, y) = 4 - x^2 - y^2$

Example If $f(x, y) = x \sin(xy)$, then

$$f_x(x, y) = xy \cos(xy) + \sin(xy)$$

and

$$f_y(x, y) = x^2 \cos(xy).$$

Example If $w = \ln(x^2 + y^2 + z^2)$, then

$$\frac{\partial w}{\partial x} = \frac{2x}{x^2 + y^2 + z^2}.$$

12.3 Higher order partial derivatives

If $f : \mathbb{R}^n \to \mathbb{R}$, then any partial derivative of f is also a function from $\mathbb{R}^n$ to $\mathbb{R}$. Hence we may evaluate partial derivatives of a partial derivative.

Notation: If $f : \mathbb{R}^2 \to \mathbb{R}$, then we may write

$$f_{xx}(x,y) = \frac{\partial^2}{\partial x^2} f(x,y),$$

$$f_{xy}(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y),$$

$$f_{yx}(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y),$$

and

$$f_{yy}(x,y) = \frac{\partial^2}{\partial y^2} f(x,y).$$

Similar notation applies to higher order derivatives and for functions of more variables.

Example If $f(x,y,z) = 4x^2yz - 8xy^2z^3$, then, for example

$$f_x(x,y,z) = 8xyz - 8y^2z^3,$$

$$f_{xy}(x,y,z) = 8xz - 16yz^3,$$

$$f_y(x,y,z) = 4x^2z - 16xyz^3,$$

$$f_{yx}(x,y,z) = 8xz - 16yz^3,$$

and

$$f_{yy}(x,y,z) = -16xz^3$$

for the second-order partial derivatives.

Note that in the previous example $f_{xy}(x,y,z) = f_{yx}(x,y,z)$. Equality of mixed partials does not hold in general, but will under the conditions of the next theorem.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$. If, for any i and k, $f_{x_i x_k}$ and $f_{x_k x_i}$ are both continuous, then

$$f_{x_i x_k}(x_1, x_2, \ldots, x_n) = f_{x_k x_i}(x_1, x_2, \ldots, x_k).$$