Lecture 11: Limits and Continuity

11.1 Limits

Definition Suppose $f : \mathbb{R}^n \to \mathbb{R}$. We say the *limit* of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is L, denoted $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(\mathbf{x}) - L| < \epsilon$$

whenever

$$0 < |\mathbf{x} - \mathbf{a}| < \delta.$$

Proposition If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = M$, and c is a scalar, then

$$\lim_{\mathbf{x}\to\mathbf{a}} cf(\mathbf{x}) = cL,$$

$$\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) + g(\mathbf{x})) = L + M,$$

$$\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) - g(\mathbf{x})) = L - M,$$

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})g(\mathbf{x}) = LM,$$

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{L}{M}, \text{ provided } M \neq 0,$$

and

$$\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}))^c = L^c.$$

Example If $f(x, y, z) = x$, then we should have

$$\lim_{(x,y,z)\to(1,2,3)} f(x, y, z) = 1.$$

To verify this, we need to show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - 1| < \epsilon$ whenever

$$|(x, y, z) - (1, 2, 3)| < \delta.$$

Now

$$|(x, y, z) - (1, 2, 3)| = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} \geq |x - 1|.$$

Hence $|x - 1| < \epsilon$ whenever

$$|(x, y, z) - (1, 2, 3)| < \epsilon.$$

That is, we may take $\delta = \epsilon$.

In general, if $f(x_1, x_2, \ldots, x_n) = x_k$, $k = 1, 2, \ldots, n$, and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, then

$$\lim_{\mathbf{x} \to \mathbf{a}} f(x_1, x_2, \ldots, x_n) = a_k.$$

Also note that for any scalar c and vector $\mathbf{a}$ in $\mathbb{R}^n$,

$$\lim_{\mathbf{x} \to \mathbf{a}} c = c.$$

A function of the form

$$f(x_1, x_2, \ldots, x_n) = c x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

where c is any scalar and m_1, m_2, ..., m_n are nonnegative integers, is called a *monomial*. A *polynomial* is a sum of monomials and a *rational function* is a ratio of polynomials.

Proposition If $f : \mathbb{R}^n \to \mathbb{R}$ is a rational function, then for any $\mathbf{a}$ in the domain of f,

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

Example $\displaystyle \lim_{(x,y) \to (3,2)} \frac{x^2 y - 4xy}{16x - 4y} = \frac{18 - 24}{48 - 8} = -\frac{6}{40} = -\frac{3}{20}.$

Recall that for a function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Similarly, for a function $f : \mathbb{R}^n \to \mathbb{R}$ we have the following: Suppose $\varphi : \mathbb{R} \to \mathbb{R}^n$ and $\alpha : \mathbb{R} \to \mathbb{R}^n$ parametrize curves in $\mathbb{R}^n$ with

$$\lim_{t \to b} \varphi(t) = \mathbf{a}$$

and

$$\lim_{t \to c} \alpha(t) = \mathbf{a}.$$

If

$$\lim_{t \to b} f(\varphi(t)) \neq \lim_{t \to c} f(\alpha(t)),$$

then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x})$ does not exist.

Example Suppose

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

Then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

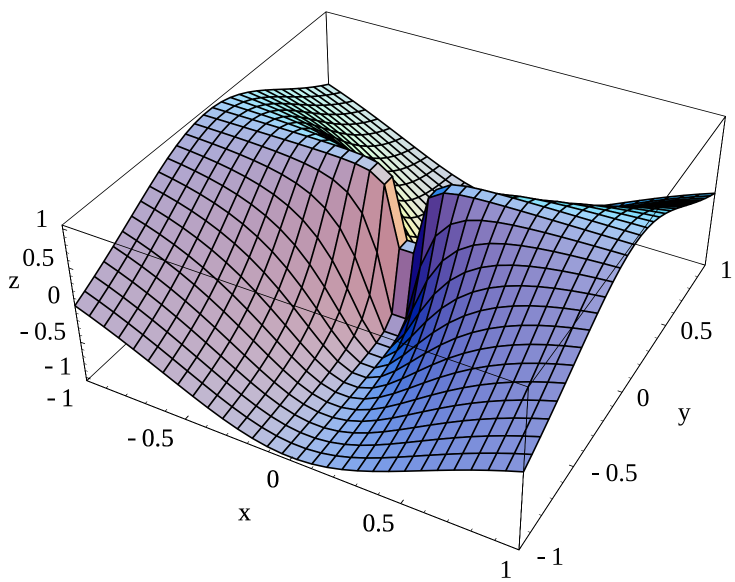
and

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{x\to 0} \frac{-y^2}{y^2} = -1,$$

so

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.



Graph of $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

Example For

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} \frac{0}{x^2} = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{x\to 0} \frac{0}{y^2} = 0,$$

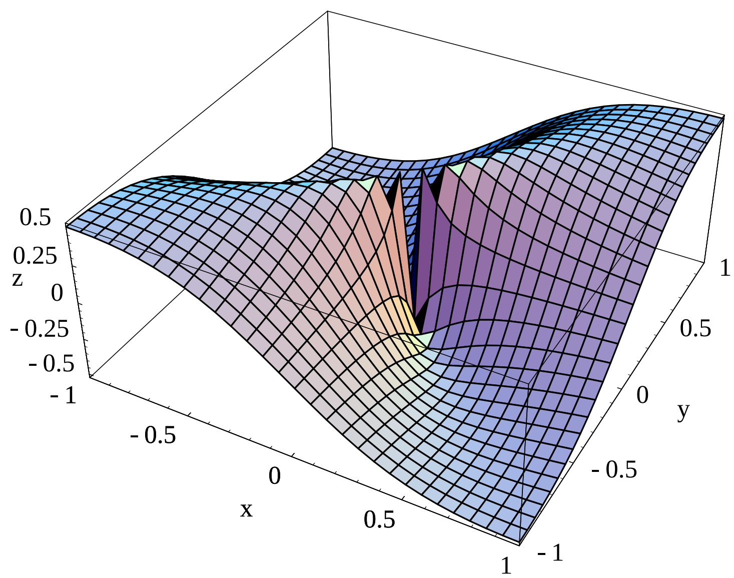
which does not help determine whether or not the limit exists. However,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2},$$

so

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.



Graph of $f(x,y) = \dfrac{xy}{x^2+y^2}$

Example For

$$f(x,y) = \frac{xy^2}{4x^2+y^4}$$

we have

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} \frac{0}{4x^2} = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = \lim_{x\to 0} \frac{0}{y^4} = 0.$$

Indeed, for any real number m,

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x,y) = \lim_{x\to 0} \frac{m^2x^3}{4x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{4+m^4x^2} = \frac{0}{4} = 0.$$

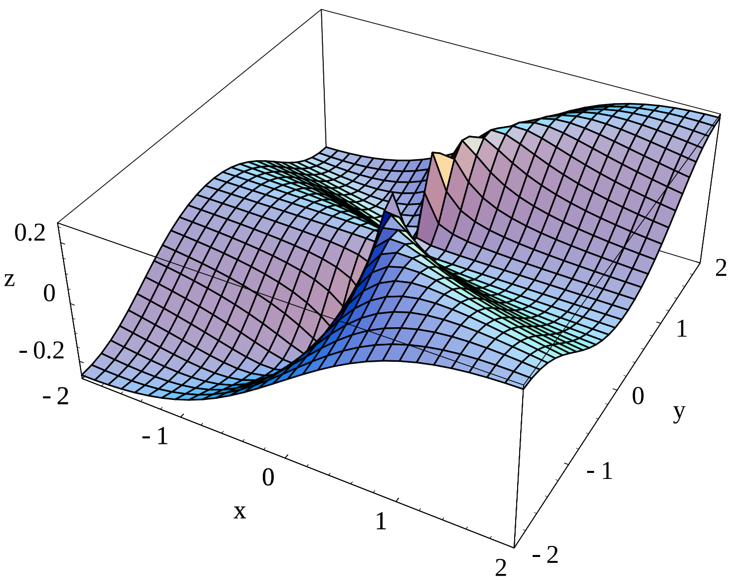
Hence the limit of $f(x, y)$ as (x, y) approaches 0 along any line through the origin is 0. However,

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} f(x,y) = \lim_{x\to 0} \frac{y^4}{5y^4} = \frac{1}{5},$$

and so

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{4x^2+y^4}$$

does not exist.



Graph of $f(x,y) = \dfrac{xy^2}{4x^2+y^4}$

It is sometimes useful in evaluating limits to make use of the fact that for any $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$,

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \geq \sqrt{x_k^2} = |x_k|$$

for any $k = 1, 2, \ldots, n$.

Example Consider

$$f(x,y) = \frac{x^2 y}{x^2+y^2}.$$

Let $\mathbf{x} = (x, y)$. Then

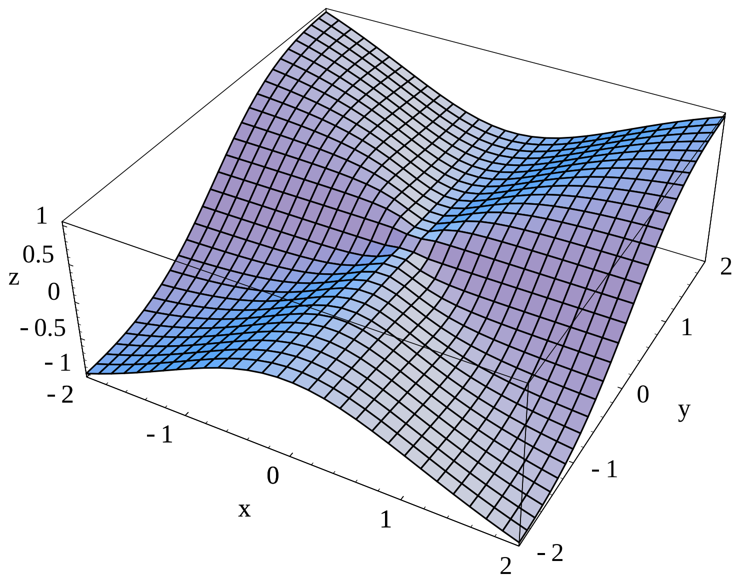
$$|f(x,y)| = \frac{|x|^2|y|}{|\mathbf{x}|^2} \leq \frac{|\mathbf{x}|^2|\mathbf{x}|}{|\mathbf{x}|^2} = |\mathbf{x}|.$$

Now

$$\lim_{(x,y)\to(0,0)} |\mathbf{x}| = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0,$$

so we must have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$$



Graph of $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$

11.2 Continuity

Definition We say a function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous* at a point $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

The following propositions follow directly from the similar proposition concerning limits.

Proposition If $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuous at $\mathbf{a}$ and c is a scalar, then the functions which take the following values at $\mathbf{x}$ are also continuous at $\mathbf{a}$:

$$f(\mathbf{x}) + g(\mathbf{x}),$$

$$f(\mathbf{x}) - g(\mathbf{x}),$$

$$cf(\mathbf{x}),$$

$$f(\mathbf{x})g(\mathbf{x}),$$

$$\frac{f(\mathbf{x})}{g(\mathbf{x})}, \text{ provided } g(\mathbf{a}) \neq 0,$$

and

$$f(\mathbf{x})^c.$$

Moreover, if $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous at $f(\mathbf{a})$, then $\varphi \circ f$ is continuous at $\mathbf{a}$.

Proposition If $f : \mathbb{R}^n \to \mathbb{R}$ is a rational function, then f is continuous at every point in its domain.

Example The function

$$f(x, y, z) = \sin(\pi\sqrt{x^2 + y^2 + z^2})$$

is continuous at every point in $\mathbb{R}^3$. We also phrase this by saying f is continuous on $\mathbb{R}^3$. For example, this means that

$$\lim_{(x,y,z)\to(1,2,2)} \sin(\pi\sqrt{x^2 + y^2 + z^2}) = \sin(\pi\sqrt{1 + 4 + 4}) = \sin(3\pi) = 0.$$