Lecture 10: Functions from $\mathbb{R}^n$ to $\mathbb{R}$

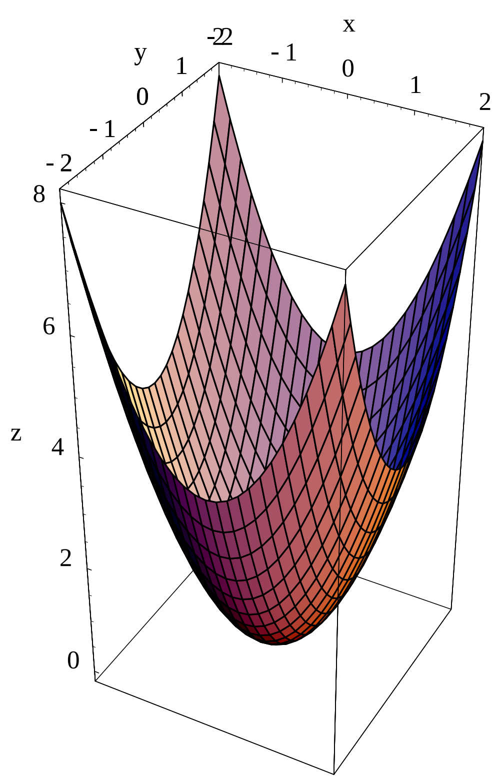
10.1 Graphs

Definition The *graph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$\{(x_1, x_2, \ldots, x_n, x_{n+1}) : x_{n+1} = f(x_1, x_2, \ldots, x_n)\}.$$

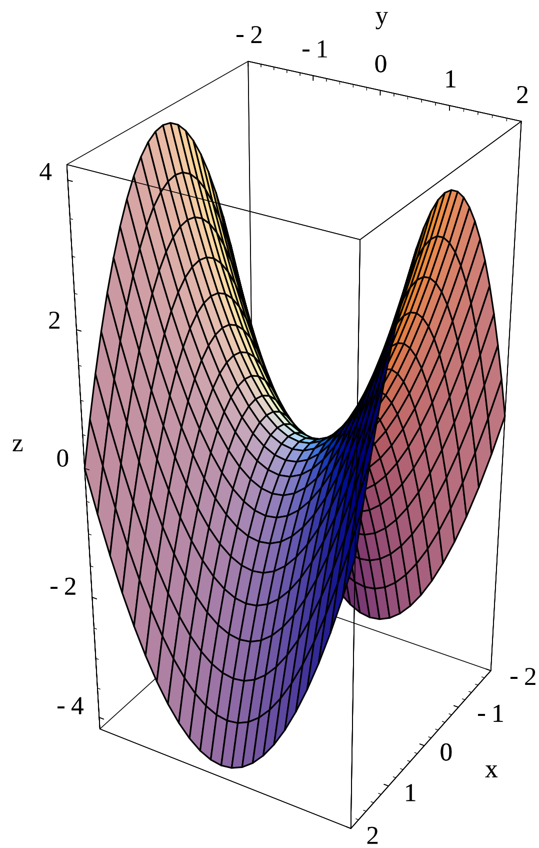
Note that if $f : \mathbb{R}^n \to \mathbb{R}$, then the graph of f is in $\mathbb{R}^{n+1}$. Hence we may visualize the graph of f only when $n = 1$, in which case the graph is a curve in $\mathbb{R}^2$, or $n = 2$, in which case the graph is a surface in $\mathbb{R}^3$.

Example The graph of $f(x, y) = x^2 + y^2$ is an example of a *paraboloid*. Note that if we fix a value of x, say $x = c$, then the curve above the line $x = c$ in the xy-plane is a parabola, namely, the graph of $z = y^2 + c^2$. Similarly, the curve above the line $y = c$ is the parabola $z = x^2 + c^2$. Moreover, if we slice the graph of f with the plane $z = c$, where $c > 0$, the resulting curve is the circle with equation $x^2 + y^2 = c$.



The graph of $f(x, y) = x^2 + y^2$

Example The graph of $f(x, y) = y^2 - x^2$ is an example of a graph with a *saddle point.* Note that above the line $x = 0$ in the xy-plane the graph is the parabola $z = y^2$, which opens upward, while above the line $x = 0$ in the xy-plane the graph is the parabola $z = -x^2$, which opens downward. The curve resulting from slicing the graph with the plane $z = c$ is the hyperbola $y^2 - x^2 = c$, which is why this graph is an example of a *hyperbolic paraboloid.*



The graph of $f(x, y) = y^2 - x^2$

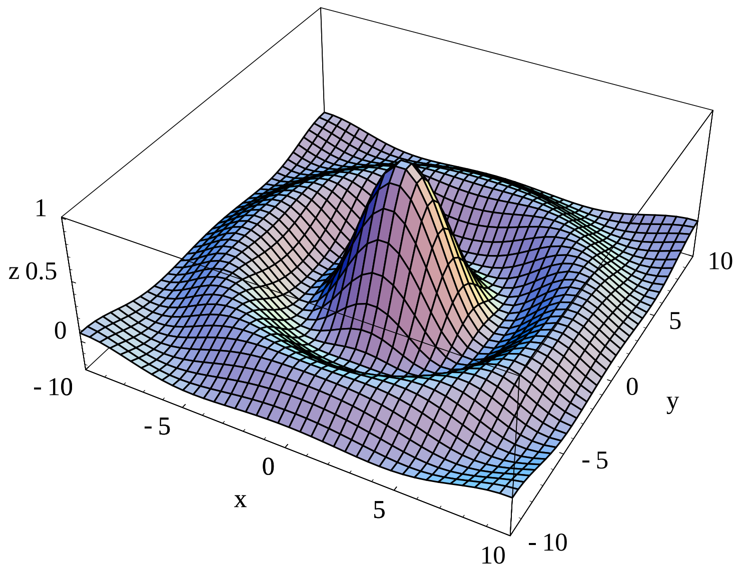
Example Suppose

$$f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}.$$

Note that the value of f at a point (x, y) depends only on the distance $r = \sqrt{x^2 + y^2}$ from (x, y) to the origin. It follows that the graph of f above any line in the xy-plane passing through the origin is the graph of

$$z = \frac{\sin(r)}{r},$$

which is a sine curve with decreasing amplitude.



The graph of $f(x,y) = \dfrac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$

10.2 Level sets

Definition If $f : \mathbb{R}^n \to \mathbb{R}$ and c is a real number, the set

$$L = \{(x_1, x_2, \ldots, x_n) : f(x_1, x_2, \ldots, x_n) = c\}$$

is called a *level set* for f.

Note that if $n = 2$, then a level set L is a curve, called a *level curve* of f, and if $n = 3$, then L is a surface, called a *level surface* of f. A plot showing numerous level curves is called a *contour plot*.

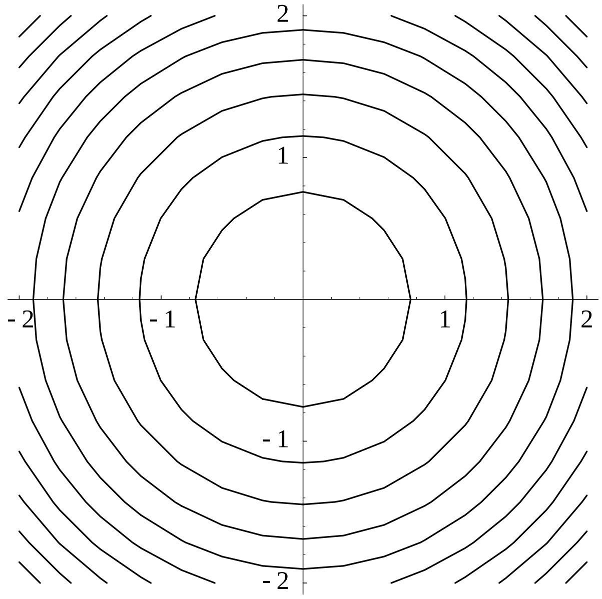
Example The level curves of $f(x,y) = x^2 + y^2$ are concentric circles with centers at the origin.

Example The graph and a contour plot of

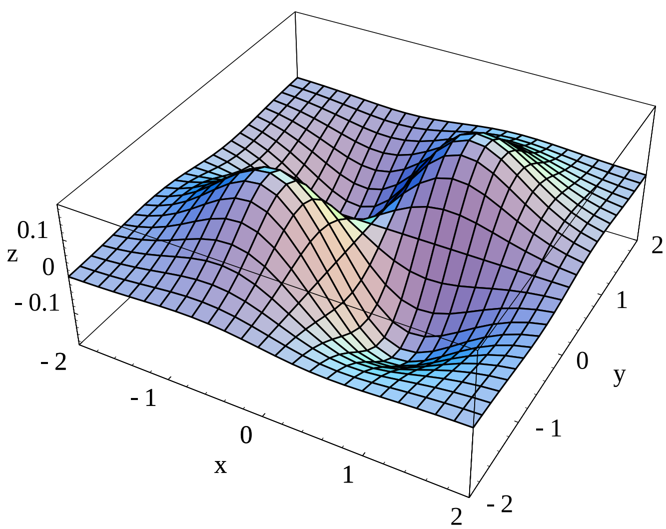
$$f(x,y) = xye^{-(x^2+y^2)}$$

are shown below.

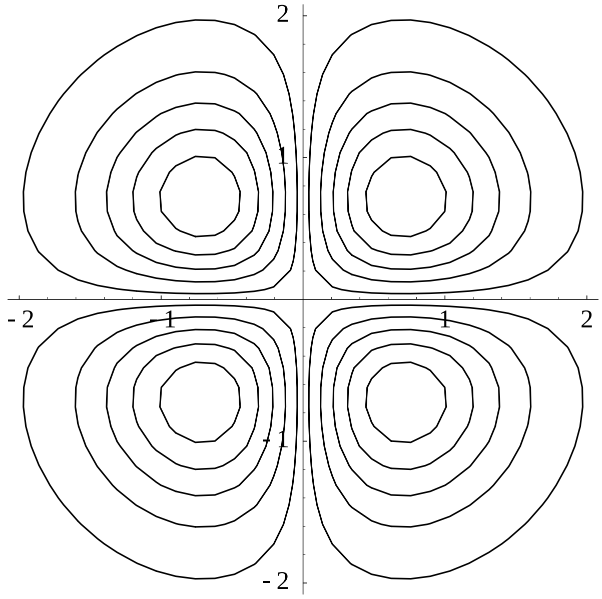
Example The level surfaces of the function $f(x,y,z) = x^2 + 2y^2 + 4z^2$ are *ellipsoids*. See the figure below for an example



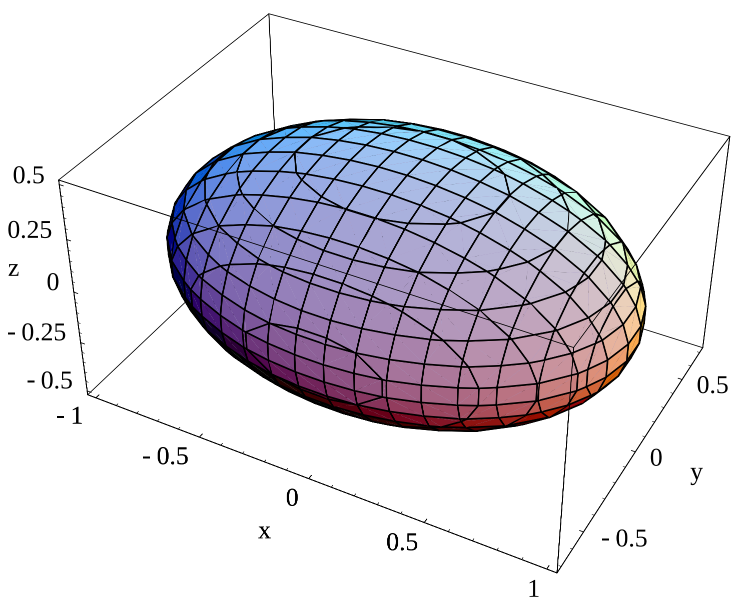
Level curves of $f(x, y) = x^2 + y^2$



The graph of $f(x, y) = xye^{-(x^2+y^2)}$



Level curves of $f(x,y) = xye^{-(x^2+y^2)}$



A level surface of $f(x,y,z) = x^2 + 2y^2 + 4z^2$