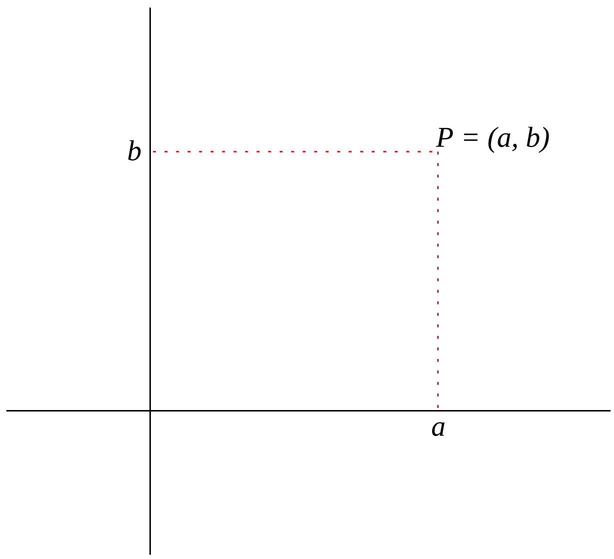
Lecture 1: Points in Space

1.1 Two and three dimensional Cartesian space

Fermat (1601 - 1665) and Descartes (1596 - 1650) had the idea of identifying points in the plane with a unique ordered pair of real numbers. That is, given a plane, we first identify two orthogonal lines, generically called the x-axis and the y-axis. If P is a point in this plane and a is the signed distance from P to the y-axis and b is the signed distance to the x-axis, then we may identify P with the ordered pair (a, b).



A point in the Cartesian plane

Similarly, for 3-space, we add a third axis, the z-axis, orthogonal to both the x-axis and the y-axis and intersecting both. Then the ordered triple (a, b, c) identifies a point P in three space located a signed distance c above the point (a, b) in the xy-plane.

Notation: We denote the points in the Cartesian plane by $\mathbb{R}^2$ and the points in Cartesian 3-space by $\mathbb{R}^3$.

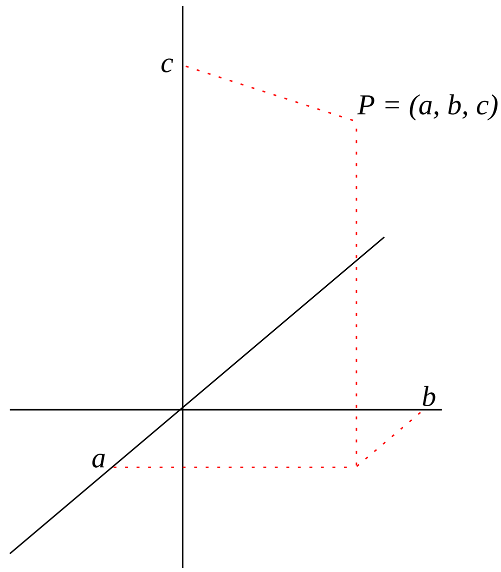
1.2 Distance

Recall that if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are points in $\mathbb{R}^2$, then the distance between P and Q is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

a consequence of the Pythagorean theorem. Now if $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are points in $\mathbb{R}^3$, then the distance between (x_1, y_1, z_1) and (x_2, y_2, z_1) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$



A point in Cartesian 3-space

and the distance from (x_2, y_2, z_1) and (x_2, y_2, z_2) is

$$|z_2 - z_1|.$$

Thus another application of the Pythagorean theorem gives us the distance from P to Q as

$$\sqrt{(\sqrt{(x_2-x_1)^2+(y_2-y_1)^2})^2+(z_2-z_1)^2} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Example The set of all points in $\mathbb{R}^3$ satisfying the equation

$$x^2 + y^2 + z^2 = 25$$

is a sphere of radius 5 centered at the origin.

In general, given $r > 0$, the set of all points in $\mathbb{R}^3$ satisfying the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

is a sphere of radius r centered at (a, b, c).

1.3 Higher dimensional spaces

Although spaces of dimensions greater than 3 are difficult to visualize, it is not hard to add coordinates to our algebraic representations. Hence we may define n-dimensional Cartesian space to be the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \text{ is a real number}, i = 1, 2, \ldots, n\}.$$

Such spaces arise in many areas of study, including physics (for example, if considering both the positions and velocities of a system of particles, or the position and time of a single particle) and economics (for example, if considering the inputs to the gross national product of a country). If $P = (x_1, x_2, \ldots, x_n)$ and $Q = (y_1, y_2, \ldots, y_n)$ are two points in $\mathbb{R}^n$, then we define the distance between P and Q to be

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

and, given $r > 0$, we call the set of all points $(x_1, x_2, \ldots, x_n)$ satisfying the equation

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = r^2$$

the $(n-1)$-dimensional sphere of radius r centered at $(a_1, a_2, \ldots, a_n)$.