Some Arithmetic

Mathematics 15: Lecture 6

Dan Sloughter

Furman University

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    of degree n has exactly (counting repetitions) n solutions in complex numbers.
  - Construction of the regular 17-gon by ruler and compass
  - Plotted the orbit of the asteroid Ceres
  - Created a non-Euclidean geometry, although he never publicly discussed it

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- More generally: $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

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    - Hence N must be prime, contradicting the assumption that p_1, p_2, p_3, ..., p_s was supposed to list all primes.
- We can still ask: how plentiful are the prime numbers? That is, how many prime numbers are there less than a given number?

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- Note: e is an irrational number which begins $e = 2.71828\ldots$.

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- The result, known as the *prime number theorem* was eventually proved by Jacques Hadamard (1865 - 1963) and Charles de la Vallée Poussin (1866 - 1962).

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- Discovered many results which anticipated the techniques of calculus

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- Gauss showed that a regular N-gon is constructable by compass and straightedge if and only if N is the product of a power of 2 and distinct Fermat primes.

Mersenne primes

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- For example,

$$63 = 2^6 - 1 = 2^{(2)(3)} - 1 = (2^3 - 1)(1 + 2^3) = 7 \times 9,$$

or

$$63 = 2^6 - 1 = 2^{(2)(3)} - 1 = (2^2 - 1)(1 + 2^2 + 2^4) = 3 \times 21.$$

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Mersenne primes (cont'd)

- Mersenne was wrong about M_{67} and M_{257}, and missed M_{61}, M_{89}, and M_{107}.
- Open question: are there an infinite number of Mersenne Primes? (See http://www.mersenne.org/ for the latest.)

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  - Are there an infinite number of perfect numbers?

Problems

1. Evaluate $1 + 2 + 3 + 4 + \cdots + 1000$.
2. Evaluate
   a. $\log_3(9)$
   b. $\log_2(64)$
   c. $\log_2\left(\frac{1}{128}\right)$
3. Approximately how many primes are there less than $1,000,000$?
4. Factor $2^{10} - 1$ and $2^{12} - 1$.
5. Use the Mersenne prime M_5 to find a perfect number.