Archimedes and *The Sand Reckoner*

Mathematics 15: Lecture 4

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September 18, 2006
Archimedes

- 287 B.C to 212 B.C.

- Used inscribed and circumscribed polygons with 96 sides to show that $\frac{223}{71} < \pi < \frac{22}{7}$.

- Found the area of a parabolic segment, using arguments that come close to those of modern calculus.

- In *The Method*, discovered in 1906, he describes his method of discovering new results, which he then proves using the strict geometrical techniques of classical Greek mathematics.

- Showed that the volume of a sphere is $\frac{2}{3}$ that of a circumscribed cylinder, and the same relation holds between their surface areas.
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Cicero and Archimedes

▶ *Cicero Discovering the Tomb of Archimedes* by Benjamin West:
Defense of Syracuse

- Did Archimedes really use mirrors to set Roman ships on fire?

See http://web.mit.edu/2.009/www/lectures/10ArchimedesResult.html
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King Hiero’s crown

Suppose

\[ w \] is the weight of the crown

\[ v_1 \] is the volume of water displaced by a weight \( w \) of gold

\[ v_2 \] is the volume of water displaced by a weight \( w \) of silver

\[ v \] is the actual volume of water displaced by the crown

\[ w_1 \] and \( w_2 \) are the actual weights of gold and silver, respectively, in the crown.

Then

\[ v = w_1 v_1 + w_2 v_2 \]

Hence

\[ vw_1 + vw_2 = v_1 w_1 + v_2 w_2 \]

And so

\[ w_1 w_2 = v_2 - v_1 \]
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- Suppose
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- $w$ is the weight of the crown
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Then

\[ v = w v_1 + w v_2 = w v_1 + w v_2 \]

Hence

\[ v w_1 + v w_2 = v_1 w_1 + v_2 w_2 \Rightarrow v w_1 + v = v_1 w_1 + v_2. \]

And so

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- Hipparchus (180 B.C. - 125 B.C.) and, later, Ptolemy (100 A.D. - 168 A.D.): planets orbit the earth in epicycles
Epicycles

- A simple epicycle:
The size of the universe

- Archimedes’ assumptions:

- Circumference of the earth < 3,000,000,000 stadia (a stadium was between 505 and 705 feet).

- Diameter of the moon < diameter of the earth < diameter of the sun.

- Diameter of the sun < 30 times the diameter of the moon (in fact, it’s more like 400 times).

- Diameter of the sun is greater than the side of a regular polygon with 1000 sides inscribed in the greatest circle in the universe (a circle with center at the center of the earth and radius extending to the center of the sun).

- Archimedes then shows that the diameter of the universe is less than 10,000,000,000 stadia (between 1 and 1.3 billion miles):

  \[ d_u < 10^3 < d_s < 10^7 < d_m < 10^{10} \times 10^6 = 10^{16}. \]
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$$d_u < \frac{1000}{3} d_s < 10,000 d_m < 10,000 d_e < 10^4 \times 10^6 = 10^{10}.$$
Note: the closest star to the earth, Alpha Centauri, is $2.48 \times 10^{13}$ miles from earth
Grains of sand

Starting with the assumptions that a poppy seed is no larger than 10,000 grains of sand and that 40 poppy seeds exceed the breadth of a finger, Archimedes now computes the number of grains of sand it would take to fill the sphere of the universe.
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- Note: he uses the fact that if you increase the diameter of a sphere by a factor of 100, then the volume of the sphere increases by a factor of $100^3 = 1,000,000$. 

He concludes that the number of grains of sand which the universe could contain is less than $10^{51}$. 

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- Note: to denote the numbers he obtains in this process, Archimedes creates a naming scheme based on $10^8 = 10,000,000,000$, which is a myriad myriad (a myriad being 10,000).
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Problems

1. If a circle of diameter \(d_1\) has area \(A_1\) and a circle of diameter \(d_2\) has area \(A_2\), show that

\[
\frac{A_1}{A_2} = \left(\frac{d_1}{d_2}\right)^2.
\]

2. If a pizza of diameter 12 inches cost $8.00, how much should a pizza of diameter 18 inches cost?

3. If a sphere of diameter \(d_1\) has volume \(V_1\) and a sphere of diameter \(d_2\) has volume \(V_2\), show that

\[
\frac{V_1}{V_2} = \left(\frac{d_1}{d_2}\right)^3.
\]

4. Given that the diameter of the earth is 4 times that of the moon, how much greater is the volume of the earth than that of the moon?
Problems (cont’d)

5. Let $A_S$ and $V_S$ be the surface area and volume, respectively, of a sphere, and let $A_C$ and $V_C$ be the surface area and volume, respectively, of a cylinder circumscribed about the sphere. Show that

$$\frac{A_S}{A_C} = \frac{V_S}{V_C} = \frac{2}{3}.$$