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Dan Sloughter (Furman University)

The Mathematician

November 30, 2006
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Crisis in foundations

- Until the 19th century, geometry was the foundation of mathematics: everything was thought to be reducible to the axioms of Euclidean geometry.

Discovery of non-Euclidean geometry:

- Euclid's parallel postulate: given a line \( \ell \) and a point \( P \) not on \( \ell \), there exists a unique line through \( P \) which is parallel to \( \ell \).

- The parallel postulate was not as empirically grounded as the other axioms.

- From Euclid on, mathematicians tried to derive the parallel postulate from the other axioms.

- In the 19th century it was realized that this was not possible: in fact, there exist geometries in which there are an infinite number of lines through \( P \) parallel to \( \ell \), and others for which there are no lines through \( P \) parallel to \( \ell \).

- The discovery of non-Euclidean geometries, along with developments in the understanding of infinities, highlighted the need for a reexamination of the foundations.
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Mathematicians needed to find a set of axioms for set theory which would be consistent, that is, contain no contradictions, and complete, that is, strong enough to prove any mathematical statement or its negation. David Hilbert (1862 - 1943) laid out a plan of attack: Specify a finite set of axioms. Specify a finite set of reasoning procedures, rules for manipulating the axioms. Demonstrate consistency and completeness using the most elementary of reasoning procedures (acceptable to all mathematicians). Hilbert's program is the beginning of the study of the procedures of mathematics, that is, meta-mathematics.
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Hilbert was willing to sacrifice meaning for consistency and completeness.
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- Any formal system which includes elementary arithmetic cannot prove its own consistency, and
- Any consistent formal system which includes elementary arithmetic is incomplete.

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The method

Gödel begins by assigning a unique number to every symbol, statement, and proof of a given symbolic language for mathematics (using the unique factorization of integers into primes), now called Gödel numbers.
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- Meta-mathematical statements, such as, “the sequence of formulas with Gödel number $x$ is a proof for the formula with Gödel number $z$,” becomes a statement concerning properties of positive integers.
- I.e., questions about mathematics become questions within mathematics.
The Gödel statement

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- We are now in the position of the liar’s paradox: if the statement is provable, then its negation is provable; if the negation of the statement is provable, then the statement is provable.

Hence, if the axioms are consistent, neither the statement nor its negation is provable.

It follows that if the axioms of mathematics are consistent, then they are incomplete.

From this basis, Gödel shows that a statement asserting consistency is one of those statements for which neither it nor its negation has a proof.
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▶ We know the Gödel statement is true, but the formal system itself cannot prove it.
▶ Hence given any formal system, there are statements which we can know to be true, even though the system cannot prove it.
▶ Roger Penrose (1931 - ), and others, have argued that this shows that there are statements which humans can know to be true that a machine can never prove (thus putting into question the possibility of artificial intelligence).
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Empiricism

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Moreover, the connection is not decisive: empirical problems do not dictate the direction of mathematical research as they do in the other sciences.
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Conclusion: There is not a priori standard of mathematical proof.
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  ▶ Note: Some mathematicians of the time recognized this, but others seemed to hold that their work was logically sound.
▶ Conclusion: There is not a priori standard of mathematical proof.
▶ So what makes a mathematical proof acceptable?
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Are we back to Hardy’s mathematics as art?