Rhind papyrus

- Written by a scribe, Ahmes, around 1700 B.C.
Rhind papyrus

- Written by a scribe, Ahmes, around 1700 B.C.
- Unclear how much is a copy of earlier text (as Ahmes himself claims), and how much is due to Ahmes
Rhind papyrus

- Written by a scribe, Ahmes, around 1700 B.C.
- Unclear how much is a copy of earlier text (as Ahmes himself claims), and how much is due to Ahmes
Part of the Rhind papyrus
Example: multiplication

Ahmes would compute $45 \times 53 = 2385$ as follows:

\[
\begin{array}{c|c}
  & 1 & 53 \\
\hline
 2 & 106 \\
\hline
 4 & 212 \\
\hline
 8 & 424 \\
\hline
16 & 848 \\
\hline
32 & 1696 \\
\hline
\text{Total} & 45 & 2385
\end{array}
\]

Note: this follows from $45 \times 53 = (1 + 4 + 8 + 32) \times 53 = 1 \times 53 + 4 \times 53 + 8 \times 53 + 32 \times 53$. 
Example: multiplication

Ahmes would compute $45 \times 53 = 2385$ as follows:

\[
\begin{array}{cc}
\backslash & 1 & 53 \\
2 & 106 \\
\backslash & 4 & 212 \\
\backslash & 8 & 424 \\
16 & 848 \\
\backslash & 32 & 1696 \\
\text{Total} & 45 & 2385 \\
\end{array}
\]

Note: this follows from

\[45 \times 53 = (1 + 4 + 8 + 32) \times 53 = 1 \times 53 + 4 \times 53 + 8 \times 53 + 32 \times 53.\]
Ahmes would compute $437 \div 23 = 19$ as follows:

\[
\begin{array}{c|c|c}
\backslash & 1 & 23 \\
\backslash & 2 & 46 \\
\backslash & 4 & 92 \\
\backslash & 8 & 184 \\
\backslash & 16 & 368 \\
\hline
\text{Total} & 19 & 437
\end{array}
\]
Example: division with remainder

Ahmes would compute $440 \div 23 = 19\frac{3}{23}$ as follows:

\[
\begin{array}{c|cc}
\backslash & 1 & 23 \\
\backslash & 2 & 46 \\
\backslash & 4 & 92 \\
\backslash & 8 & 184 \\
\backslash & 16 & 368 \\
Total & 19 & 437 \\
\end{array}
\]
Example: division with remainder

- Ahmes would compute $440 \div 23 = 19 \frac{3}{23}$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>46</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>92</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>184</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>368</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>19</td>
<td>437</td>
</tr>
</tbody>
</table>

- But: Egyptians insisted on fractions with unit numerator (except for $\frac{2}{3}$).
Fractions

Example: instead of $\frac{3}{4}$, Egyptians would write $\frac{1}{2}, \frac{1}{4}$. 
Fractions

Example: instead of $\frac{3}{4}$, Egyptians would write $\frac{1}{2}$, $\frac{1}{4}$.

Example: $\frac{2}{61}$ was written as $\frac{1}{40}$, $\frac{1}{244}$, $\frac{1}{488}$, $\frac{1}{610}$. Why not as $\frac{1}{61}$, $\frac{1}{61}$?
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:

Let \( y \) be the smallest share and let \( x \) be the difference in shares.

Then the shares are:

\[ y, \quad y + x, \quad y + 2x, \quad y + 3x, \quad y + 4x. \]

So we want:

\[ 100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x \]

and

\[ 2y + x = 3y + 9x \times \frac{1}{7}. \]

Hence \( 14y + 7x = 3y + 9x \), and so \( 11y = 2x \).

Substituting \( x = \frac{11}{2}y \) into the first equation, we have

\[ 100 = 5y + 55y = 60y. \]

Hence \( y = \frac{5}{3} \), \( x = \frac{55}{6} \), and the shares are

\[ \frac{5}{3}, \quad \frac{65}{6}, \quad \frac{20}{3}, \quad \frac{175}{6}, \quad \frac{115}{3}. \]
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:

Let \( y \) be the smallest share and let \( x \) be the difference in shares.

Then the shares are: 
- \( y \)
- \( y + x \)
- \( y + 2x \)
- \( y + 3x \)
- \( y + 4x \)

So we want:

\[
100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x
\]

and

\[
\frac{2y + x}{7} = \frac{3y + 9x}{7}
\]

Hence

\[
14y + 7x = 3y + 9x
\]

and so

\[
11y = 2x
\]

Substituting \( x = \frac{11}{2}y \) into the first equation, we have

\[
100 = 5y + 55y = 60y
\]

Hence

\[
y = \frac{5}{3}, x = \frac{55}{6}
\]

and the shares are:
- \( \frac{5}{3} \)
- \( 6\frac{5}{6} \)
- \( 20 \)
- \( 17\frac{5}{6} \)
- \( 11\frac{5}{3} \)
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:

Let $y$ be the smallest share and let $x$ be the difference in shares.

Then the shares are: $y$, $y + x$, $y + 2x$, $y + 3x$, and $y + 4x$.

So we want:

$100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x$ and

$2y + x = 3y + 9x$.

Hence $14y + 7x = 3y + 9x$, and so $11y = 2x$.

Substituting $x = \frac{11}{2}y$ into the first equation, we have $100 = 5y + 55y = 60y$.

Hence $y = \frac{5}{3}$, $x = \frac{55}{6}$, and the shares are $\frac{5}{3}$, $\frac{65}{6}$, $20$, $\frac{175}{6}$, $\frac{115}{3}$.
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:

Let $y$ be the smallest share and let $x$ be the difference in shares.

Then the shares are: $y$, $y + x$, $y + 2x$, $y + 3x$, and $y + 4x$. 

So we want:

$$100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x$$

and

$$2y + x = 3y + 9x \quad \text{on multiplying by 7}.$$

Hence $14y + 7x = 3y + 9x$, and so $11y = 2x$.

Substituting $x = \frac{11}{2}y$ into the first equation, we have

$$100 = 5y + 55y = 60y.$$ 

Hence $y = \frac{5}{3}$, $x = \frac{55}{6}$, and the shares are $\frac{5}{3}$, $\frac{65}{6}$, $\frac{20}{6}$, $\frac{175}{6}$, $\frac{115}{3}$. 
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:

Let $y$ be the smallest share and let $x$ be the difference in shares.
Then the shares are: $y$, $y + x$, $y + 2x$, $y + 3x$, and $y + 4x$.

So we want:

$$100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x$$

and

$$2y + x = \frac{3y + 9x}{7}.$$
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:

Let $y$ be the smallest share and let $x$ be the difference in shares.

Then the shares are: $y, y + x, y + 2x, y + 3x,$ and $y + 4x.$

So we want:

\[
100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x
\]

and

\[
2y + x = \frac{3y + 9x}{7}.
\]

Hence $14y + 7x = 3y + 9x,$ and so $11y = 2x.$
The loaves problem

Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

Modern algebraic solution:
- Let $y$ be the smallest share and let $x$ be the difference in shares.
- Then the shares are: $y, y + x, y + 2x, y + 3x,$ and $y + 4x$.
- So we want:

$$100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x$$

and

$$2y + x = \frac{3y + 9x}{7}.$$

- Hence $14y + 7x = 3y + 9x$, and so $11y = 2x$.
- Substituting $x = \frac{11}{2}y$ into the first equation, we have

$$100 = 5y + 55y = 60y.$$
The loaves problem

- Problem: Divide 100 loaves between 5 men so that the shares are in arithmetical progression and the sum of the smallest two shares is one-seventh of the sum of the three largest shares.

- Modern algebraic solution:
  - Let $y$ be the smallest share and let $x$ be the difference in shares.
  - Then the shares are: $y$, $y + x$, $y + 2x$, $y + 3x$, and $y + 4x$.
  - So we want:
    
    $$100 = y + (y + x) + (y + 2x) + (y + 3x) + (y + 4x) = 5y + 10x$$

    and

    $$2y + x = \frac{3y + 9x}{7}.$$

  - Hence $14y + 7x = 3y + 9x$, and so $11y = 2x$.
  - Substituting $x = \frac{11}{2}y$ into the first equation, we have
    
    $$100 = 5y + 55y = 60y.$$

  - Hence $y = \frac{5}{3}$, $x = \frac{55}{6}$, and the shares are
    
    $$\frac{5}{3}, \frac{65}{6}, 20, \frac{175}{6}, \frac{115}{3}.$$
Solution using \textit{false position}: 

Let the smallest share be 1 loaf and let $x$ be the difference in shares. That is, the shares are 1, 1 + $x$, 1 + 2$x$, 1 + 3$x$, and 1 + 4$x$.

Then we want $2 + x = 3 + 9x$. Hence $14 + 7x = 3 + 9x$, so $11 = 2x$ and $x = \frac{11}{2}$.

Now the sum of the shares is $5 + 10x = 60$, but we want it to be 100. So multiply by $\frac{100}{60} = \frac{5}{3}$.

Thus the shares are $\frac{5}{3}$, $\frac{65}{6}$, 20, $\frac{175}{6}$, and $\frac{115}{3}$.
The loaves problem (cont’d)

- Solution using *false position*:
  - Let the smallest share be 1 loaf and let $x$ be the difference in shares.
Solution using false position:

- Let the smallest share be 1 loaf and let $x$ be the difference in shares.
- That is, the shares are $1$, $1 + x$, $1 + 2x$, $1 + 3x$, and $1 + 4x$. 

Then we want $2 + x = 3 + 9x$. Hence $14 + 7x = 3 + 9x$, so $11 = 2x$ and $x = \frac{11}{2}$.

Now the sum of the shares is $5 + 10x = 60$, but we want it to be 100. So multiply by $\frac{100}{60} = \frac{5}{3}$.

Thus the shares are $\frac{5}{3}$, $\frac{65}{6}$, 20, $\frac{175}{6}$, and $\frac{115}{3}$. 
Solution using false position:

- Let the smallest share be 1 loaf and let $x$ be the difference in shares.
- That is, the shares are $1, 1 + x, 1 + 2x, 1 + 3x, \text{ and } 1 + 4x$.
- Then we want

$$2 + x = \frac{3 + 9x}{7}.$$
Solution using *false position*:

- Let the smallest share be 1 loaf and let \( x \) be the difference in shares.
- That is, the shares are 1, 1 + \( x \), 1 + 2\( x \), 1 + 3\( x \), and 1 + 4\( x \).
- Then we want
  \[
  2 + x = \frac{3 + 9x}{7}.
  \]
- Hence 14 + 7\( x \) = 3 + 9\( x \), so 11 = 2\( x \) and \( x = \frac{11}{2} \).
Solution using false position:

- Let the smallest share be 1 loaf and let $x$ be the difference in shares.
- That is, the shares are 1, $1 + x$, $1 + 2x$, $1 + 3x$, and $1 + 4x$.
- Then we want
  \[ 2 + x = \frac{3 + 9x}{7}. \]
- Hence $14 + 7x = 3 + 9x$, so $11 = 2x$ and $x = \frac{11}{2}$.
- Now the sum of the shares is $5 + 10x = 60$, but we want it to be 100. So multiply by $\frac{100}{60} = \frac{5}{3}$.
Solution using *false position*:

Let the smallest share be 1 loaf and let $x$ be the difference in shares. That is, the shares are $1, 1 + x, 1 + 2x, 1 + 3x, \text{ and } 1 + 4x$.

Then we want

$$2 + x = \frac{3 + 9x}{7}.$$ 

Hence $14 + 7x = 3 + 9x$, so $11 = 2x$ and $x = \frac{11}{2}$.

Now the sum of the shares is $5 + 10x = 60$, but we want it to be 100. So multiply by $\frac{100}{60} = \frac{5}{3}$.

Thus the shares are $\frac{5}{3}, \frac{65}{6}, 20, \frac{175}{6}, \text{ and } \frac{115}{3}$.
Ahmes may have found $x$ by trial and error:

- If $x = 1$, then $2 + x = 3$ and $3 + 9x = 12$, for a difference of $9$.
- If $x = 2$, then $2 + x = 4$ and $3 + 9x = 3$, for a difference of $1$.
- Hence increasing $x$ by 1 decreases the difference by $2$.
- Since we want to decrease the difference by $9$, we should increase $x$ by $\frac{9}{2} = 9$.
- Hence $\frac{11}{2}$ should work.
Ahmes may have found $x$ by trial and error:

- If $x = 1$, then $2 + x = 3$ and $\frac{3+9x}{7} = \frac{12}{7}$, for a difference of $\frac{9}{7}$.
- Increasing $x$ by 1 decreases the difference by $\frac{2}{7}$.
- Since we want to decrease the difference by $\frac{9}{7}$, we should increase $x$ by $\frac{9}{7} \div \frac{2}{7} = \frac{9}{2}$.
- Hence $\frac{11}{2}$ should work.
Ahmes may have found $x$ by trial and error:

- If $x = 1$, then $2 + x = 3$ and $\frac{3 + 9x}{7} = \frac{12}{7}$, for a difference of $\frac{9}{7}$.
- If $x = 2$, then $2 + x = 4$ and $\frac{3 + 9x}{7} = 3$, for a difference of 1.
Ahmes may have found $x$ by trial and error:

- If $x = 1$, then $2 + x = 3$ and $\frac{3 + 9x}{7} = \frac{12}{7}$, for a difference of $\frac{9}{7}$.
- If $x = 2$, then $2 + x = 4$ and $\frac{3 + 9x}{7} = 3$, for a difference of 1.
- Hence increasing $x$ by 1 decreases the difference by $\frac{2}{7}$.

Since we want to decrease the difference by $\frac{9}{7}$, we should increase $x$ by $\frac{9}{7}$.

Hence $\frac{11}{2}$ should work.
Ahmes may have found $x$ by trial and error:

- If $x = 1$, then $2 + x = 3$ and $\frac{3 + 9x}{7} = \frac{12}{7}$, for a difference of $\frac{9}{7}$.
- If $x = 2$, then $2 + x = 4$ and $\frac{3 + 9x}{7} = 3$, for a difference of 1.
- Hence increasing $x$ by 1 decreases the difference by $\frac{2}{7}$.
- Since we want to decrease the difference by $\frac{9}{7}$, we should increase $x$ by

$$\frac{\frac{9}{7}}{\frac{2}{7}} = \frac{9}{2}.$$
Ahmes may have found $x$ by trial and error:

- If $x = 1$, then $2 + x = 3$ and $\frac{3 + 9x}{7} = \frac{12}{7}$, for a difference of $\frac{9}{7}$.
- If $x = 2$, then $2 + x = 4$ and $\frac{3 + 9x}{7} = 3$, for a difference of 1.
- Hence increasing $x$ by 1 decreases the difference by $\frac{2}{7}$.
- Since we want to decrease the difference by $\frac{9}{7}$, we should increase $x$ by

$$\frac{9}{7} \div \frac{2}{7} = \frac{9}{2}.$$

- Hence $\frac{11}{2}$ should work.
Numbers

- How important is the way we write numbers?
Numbers

- How important is the way we write numbers?
- What objects are allowed as numbers?

- **Natural numbers** (or **positive integers**) \(1, 2, 3, \ldots\)
- **Non-negative integers** \(0, 1, 2, 3, \ldots\)
- **Integers**: \(\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\)
- **Rational numbers**: set of all ratios \(\frac{a}{b}\), where \(a \text{ and } b\) are integers, \(b \neq 0\).
- **Real numbers**: set of all rational and irrational numbers (such as \(\sqrt{2}\) and \(\pi\)).
Numbers

- How important is the way we write numbers?
- What objects are allowed as numbers?
  - *Natural numbers, or positive integers*: 1, 2, 3, …
How important is the way we write numbers?

What objects are allowed as numbers?

- **Natural numbers**, or **positive integers**: 1, 2, 3, ...
- **Non-negative integers**: 0, 1, 2, 3, ...

**Rational numbers**: set of all ratios \( \frac{a}{b} \), where \( a \) and \( b \) are integers, \( b \neq 0 \).

**Real numbers**: set of all rational and **irrational** numbers (such as \( \sqrt{2} \) and \( \pi \)).
Numbers

- How important is the way we write numbers?
- What objects are allowed as numbers?
  - *Natural numbers*, or *positive integers*: 1, 2, 3, ...  
  - *Non-negative integers*: 0, 1, 2, 3, ...  
  - *Integers*: ..., −3, −2, −1, 0, 1, 2, 3, ...

Rational numbers: set of all ratios \( \frac{a}{b} \), where \( a \) and \( b \) are integers, \( b \neq 0 \).

Real numbers: set of all rational and irrational numbers (such as \( \sqrt{2} \) and \( \pi \)).
Numbers

- How important is the way we write numbers?
- What objects are allowed as numbers?
  - *Natural numbers*, or *positive integers*: $1, 2, 3, \ldots$
  - *Non-negative integers*: $0, 1, 2, 3, \ldots$
  - *Integers*: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
  - *Rational numbers*: set of all ratios $\frac{a}{b}$, where $a$ and $b$ are integers, $b \neq 0$. 

Dan Sloughter (Furman University)  
The Rhind Papyrus  
September 13, 2006
Numbers

- How important is the way we write numbers?
- What objects are allowed as numbers?
  - *Natural numbers*, or *positive integers*: 1, 2, 3, …
  - *Non-negative integers*: 0, 1, 2, 3, …
  - *Integers*: …, −3, −2, −1, 0, 1, 2, 3, …
  - *Rational numbers*: set of all ratios $\frac{a}{b}$, where $a$ and $b$ are integers, $b \neq 0$.
  - *Real numbers*: set of all rational and *irrational* numbers (such as $\sqrt{2}$ and $\pi$)
Problems

1. Compute $25 \times 35$ using the Egyptian method.
2. Compute $414 \div 18$ using the Egyptian method.
3. The Rhind papyrus appears to calculate the area of a circle as follows: Draw a square with each side 9 units in length. Divide the sides of the square in thirds and inscribe an octagon in the square by cutting off the corners of the square as shown in the figure below.

![Octagon and Circle Diagram]

a. Show that the area of the octagon is 63 square units.

b. Notice that the area of the circle inscribed in the square is close to the area of the octagon, perhaps slightly larger. Hence we might approximate the area of the circle by 64 square units. How close is this to the exact area of the circle?
3. Continuation:
   
c. The previous observation led to the conclusion that the area of a circle of diameter \(d\) is well approximated by the area of a square with each side of length \(\frac{8}{9} d\). Explain this conclusion.

d. Show that saying that the area of a circle is equal to the area of a square with sides of length eight-ninths the diameter of the circle is equivalent to taking \(\pi\) equal to 

\[
\left(\frac{16}{9}\right)^2 \approx 3.1605.
\]