Mathematics of a Lady Tasting Tea

Mathematics 15: Lecture 19

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- Equally famous as a geneticist (for example, the text *The Genetical Theory of Natural Selection*)
- Refused a prestigious position in London to pursue statistical problems in agriculture at Rothamsted, where he developed, among many other fundamental notions of modern statistics, the theory of randomized experimental design

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- To test her claim, eight cups of tea are prepared, four of which have the milk added first and four of which have the tea added first.
- Question: How many cups does she have to correctly identify to convince us of her ability?

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- For example, the probability that she would correctly identify all 4 cups is $\frac{1}{70}$.

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- Since she can now choose the 1 incorrect cup 4 ways, there are a total of $4 \times 4 = 16$ ways for her to choose exactly 3 right and 1 wrong.
- Hence the probability that she chooses exactly 3 correctly is $\frac{16}{70} = \frac{8}{35}$.

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- Note: typically, a result is considered statistically significant if the probability of its occurrence is less than 0.05, that is, less than 1 out of 20.

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- Note: in most situations, rejecting the null hypothesis is what we hope to do.

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- Randomization takes care of all the possible causes for which we cannot control.

Problems

1. Suppose the lady samples 10 cups of tea, among which 5 had the tea poured first and 5 had the milk poured first.
   a. What is the probability she correctly identifies all five cups which had the tea poured first?
   b. What is the probability she correctly identifies exactly four of the cups which had the tea poured first?
   c. What is the probability she correctly identifies four or more of the cups which had the tea poured first?
   d. Would we reject the null hypothesis if she correctly identified exactly four of the cups which had the tea poured first?