Lecture 9: Trigonometric Substitutions

9.1 Sine substitutions

We have seen previously that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c.$$

However, we know this result only as a consequence of our result for differentiating the arcsine function. To obtain the value of this integral directly, we could begin with the substitution

$$\begin{aligned} x &= \sin(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= \cos(u)du. \end{aligned}$$

Note that the restriction on u ensures that u is in the range of the arcsine function and that $\sin(u)$ is in the domain of the function we are integrating. Moreover, note that we have defined x as a function of u rather than our standard form of substitution in which we define u as a function of x. We could have defined the substitution in the other order, namely, setting $u = \sin^{-1}(x)$, but it is more convenient for our purposes now to use $x = \sin(x)$. With this substitution, we have

$$\begin{aligned} \int \frac{1}{\sqrt{1-x^2}}\,dx &= \int \frac{\cos(u)}{\sqrt{1-\sin^2(u)}}\,du \\ &= \int \frac{\cos(u)}{\sqrt{\cos^2(u)}}\,du \\ &= \int \frac{\cos(u)}{\cos(u)}\,du \\ &= \int du \\ &= u + c \\ &= \sin^{-1}(x) + c. \end{aligned}$$

Note that we used the fact that $\cos(u) \geq 0$, which follows from the condition $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

In general, for an integral involving $\sqrt{a^2 - x^2}$, where $a > 0$, the substitution $x = a\sin^{-1}(u)$ for $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ will yield the simplification

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2(u)} = \sqrt{a^2\cos^2(u)} = a\cos(u).$$

Example To evaluate $\int \sqrt{4 - x^2}dx$, we use the substitution

$$\begin{aligned} x &= 2\sin(u), -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \\ dx &= 2\cos(u)du. \end{aligned}$$

Then we have

$$\begin{aligned}\int \sqrt{4-x^2}dx &= \int \sqrt{4-4\sin^2(u)}(2\cos(u))du \\ &= 4\int \cos^2(u)du \\ &= 2\int (1+\cos(2u))du \\ &= 2u+\sin(2u)+c \\ &= 2\sin^{-1}\left(\frac{x}{2}\right)+2\sin(u)\cos(u)+c \\ &= 2\sin^{-1}\left(\frac{x}{2}\right)+\frac{x\sqrt{4-x^2}}{2}+c.\end{aligned}$$

If we are evaluating a definite integral, changing the limits of integration will make the work of changing back to the original variable of integration unnecessary.

Example Using that same substitution as the previous example, we have

$$\int_0^2 \sqrt{4-x^2}dx = 4\int_0^{\frac{\pi}{2}} \cos^2(u)du = 2u\Big|_0^{\frac{\pi}{2}} + \sin(2u)\Big|_0^{\frac{\pi}{2}} = \pi.$$

9.2 Tangent substitutions

For an integral involving a^2+x^2 or $\sqrt{a^2+x^2}$, the substitution $x = a\tan(u)$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$, will yield the simplification

$$a^2+x^2 = a^2+a^2\tan^2(u) = a^2\sec^2(u).$$

Example To evaluate $\displaystyle\int_0^4 \frac{1}{16+x^2}\,dx$, we use the substitution

$$\begin{aligned} x &= 4\tan(x), \\ dx &= 4\sec^2(x)dx.\end{aligned}$$

Then we have

$$\begin{aligned}\int_0^4 \frac{1}{16+x^2}\,dx &= \int_0^{\frac{\pi}{4}} \frac{4\sec^2(u)}{16+16\tan^2(u)}\,du \\ &= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{\sec^2(u)}{1+\tan^2)u)}\,du \\ &= \frac{1}{4}\int_u^{\frac{\pi}{4}} du \\ &= \frac{\pi}{16}.\end{aligned}$$

Example To evaluate $\int \frac{1}{\sqrt{4+x^2}}\,dx$, we use the substitution

$$\begin{aligned} x &= 2\tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= 2\sec^2(u)du. \end{aligned}$$

Then we have

$$\begin{aligned} \int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{2\sec^2(u)}{\sqrt{4+4\tan^2(u)}}\,du \\ &= \int \frac{\sec^2(u)}{\sqrt{\sec^2(u)}}\,du \\ &= \int \sec(u)du \\ &= \log|\sec(u)+\tan(u)| + c \\ &= \log\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c \\ &= \log|\sqrt{4+x^2}+x| - \log(2) + c \\ &= \log|\sqrt{4+x^2}+x| + c. \end{aligned}$$

9.3 Secant substitutions

For an integral involving $\sqrt{x^2-a^2}$, where $a > 0$, the substitution $x = a\sec(u)$, $0 \le u < \frac{\pi}{2}$ or $\pi \le u < \frac{3\pi}{2}$, will yield the simplification

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2(u)-a^2} = a\sqrt{\sec^2(u)-1} = a\sqrt{\tan^2(u)} = a\tan(u).$$

Note that we used the fact that $\tan(u) \ge 0$ for the possible values of u.

Example To evaluate $\int \frac{1}{\sqrt{x^2-9}}\,du$, we use the substitution

$$\begin{aligned} x &= 3\sec(u), 0 \le u < \frac{\pi}{2} \text{ or } \pi \le u < \frac{3\pi}{2}, \\ du &= 3\sec(u)\tan(u)du. \end{aligned}$$

Then we have

$$\int \frac{1}{\sqrt{x^2-9}}\,du = \int \frac{3\sec(u)\tan(u)}{\sqrt{9\sec^2(u)-9}}\,du$$

$$
\begin{aligned}
&= \int \frac{\sec(u)\tan(u)}{\sqrt{\tan^2(x)}}\,du \\
&= \int \sec(u)du \\
&= \log|\sec(u)+\tan(u)| + c \\
&= \log\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + c \\
&= \log|x+\sqrt{x^2-9}| - \log(3) + c \\
&= \log|x+\sqrt{x^2-9}| + c.
\end{aligned}
$$

9.4 Completing the square

Example To evaluate $\int_{-\frac{1}{2}}^{0} \frac{1}{\sqrt{x^2+x+1}}\,dx$, we first complete the square

$$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}.$$

The substitution $u = x + \frac{1}{2}$ then gives us

$$\int_{-\frac{1}{2}}^{0} \frac{1}{\sqrt{x^2+x+1}}\,dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{u^2+\frac{3}{4}}}\,du.$$

Now let

$$
\begin{aligned}
u &= \frac{\sqrt{3}}{2}\tan(\theta), \\
du &= \frac{\sqrt{3}}{2}\sec^2(\theta)d\theta,
\end{aligned}
$$

which give us

$$
\begin{aligned}
\int_{-\frac{1}{2}}^{0} \frac{1}{\sqrt{x^2+x+1}}\,dx &= \int_0^{\frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2}\sec^2(\theta)}{\sqrt{\frac{3}{4}\tan^2(\theta)+\frac{3}{4}}}\,d\theta \\
&= \int_0^{\frac{\pi}{6}} \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta \\
&= \int_0^{\frac{\pi}{6}} \sec(\theta)d\theta \\
&= \log|\sec(\theta)+\tan(\theta)|\Big|_0^{\frac{\pi}{6}}
\end{aligned}
$$

$$
\begin{aligned}
&= \log\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) \\
&= \log(\sqrt{3}) \\
&= \frac{\log(3)}{2}.
\end{aligned}
$$