Lecture 1: Inverse Functions

1.1 Inverse Functions

Definition Suppose f is a function with domain S and range T. If g is a function with domain T and range S with the property that $f(g(x)) = x$ for every x in S and $g(f(x)) = x$ for every x in T, then we call g the *inverse* of f.

Example Let $f(x) = 3x + 2$ and $g(x) = \frac{1}{3}(x - 2)$. Then for any real number x we have

$$f(g(x)) = f\left(\frac{1}{3}(x-2)\right) = 3\left(\frac{1}{3}(x-2)\right) + 2 = x$$

and

$$g(f(x)) = g(3x+2) = \frac{1}{3}((3x+2)-2) = x.$$

Hence g is the inverse of f.

Note that if g is the inverse of f, then f is also the inverse of g.

Notation: We often denote the inverse of f by f^{-1}. For example, for the previous example we could write

$$f^{-1}(x) = \frac{1}{3}(x-2).$$

Example Let $f(x) = \sqrt{x-3}$. Note that f has domain $[3, \infty)$ and range $[0, \infty)$. To find an inverse g for f, we set $x = f(y)$ and solve for y. That is, we let

$$x = \sqrt{y-3},$$

and solve for y to find

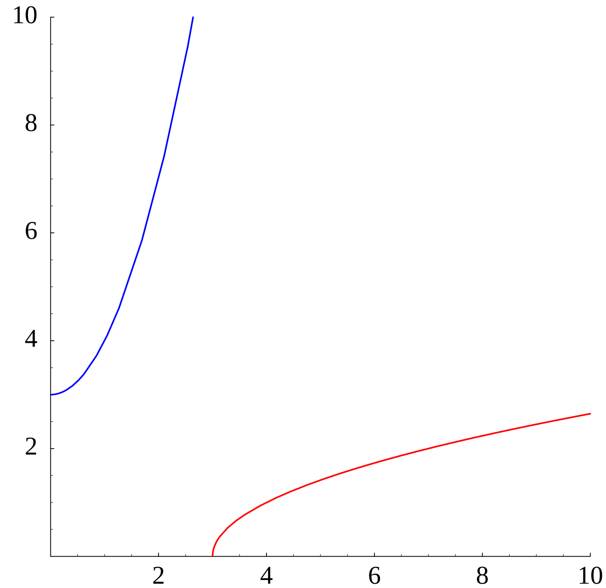
$$y = x^2 + 3.$$

Hence the function $g(x) = x^2 + 3$, with domain $[0, \infty)$, is the inverse of f. The graphs of f and g are shown below.

Note that in the previous example the graph of g is the graph of f reflected about the line $y = x$. This relationship always holds between the graph of f and its inverse (since if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of f^{-1}).

1.2 One-to-one functions

The function $f(x) = x^2$ does not have an inverse because both $f(2) = 4$ and $f(-2) = 4$. That is, an inverse function would need to send 4 to both 2 and -2, which behavior is not



Graphs of $f(x) = \sqrt{x-3}$ and $g(x) - x^2 + 3$

allowed for a function. Hence a function f has an inverse if and only if for every value y in the range of f there is a unique value x in the range of f for which $f(x) = y$.

Definition Suppose f is a function with the property that for every u and v in the domain of f we have $f(u) = f(v)$ implies that $u = v$. Then we say f is *one-to-one*.

Note that if f is increasing on an interval (a, b), then f is one-to-one and so has an inverse on (a, b). Similarly, if f is decreasing on (a, b), then f has in inverse on (a, b).

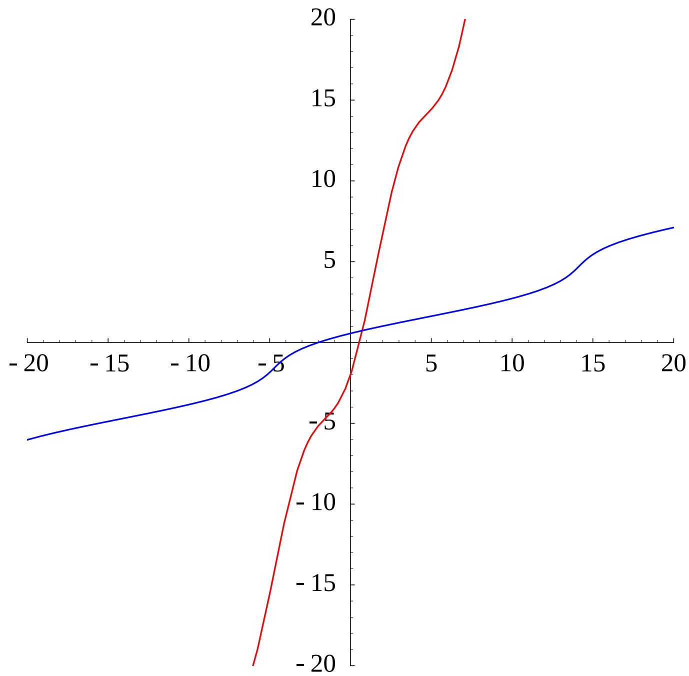
Example Although $f(x) = x^2$ does not have in inverse on all of $(-\infty, \infty)$, it is increasing on $[0, \infty)$ and so has an inverse if restricted to this interval (namely, $f^{-1}(x) = \sqrt{x}$). Similarly, if we restrict f to the interval $(-\infty, 0]$, then f is decreasing and so has an inverse (namely, $f^{-1}(x) = -\sqrt{x}$).

Proposition If f is differentiable on (a, b) with either $f'(x) > 0$ or $f'(x) < 0$ for all x in (a, b), then f has an inverse on (a, b).

Example Let $f(x) = 3x - 2\cos(x)$. Then $f'(x) = 3 + 2\sin(x)$, and so $f'(x) > 0$ for all x in $(-\infty, \infty)$. Hence f has an inverse on $(-\infty, \infty)$. The graphs of f and its inverse are shown below.

1.3 Derivatives of inverse functions

Suppose f is differentiable with inverse g and $f(s) = t$. Moreover, suppose g is differentiable at t and $f'(s) \neq 0$. Then $f(g(x)) = x$ for all x, so



Graph of $f(x) = 3x - 2\cos(x)$ and its inverse

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx} x.$$

Hence

$$f'(g(x))g'(x) = 1.$$

In particular,

$$g'(t) = \frac{1}{f'(g(t))} = \frac{1}{f'(s)}.$$

Of course, this is exactly what we should expect given the relationship between the graphs of f and g.

More generally, it is possible to prove the following theorem.

Proposition Suppose f is differentiable and one-to-one on the interval (a, b) with range (c, d). If $f'(x) \neq 0$ for all x in (a, b), then the inverse g of f is differentiable on (c, d). Moreover, if $t = f(s)$, then

$$g'(t) = \frac{1}{f'(s)}.$$

Example If $f(x) = 3x - 2\cos(x)$, then $f(0) = -2$. Hence, if $g = f^{-1}$,

$$g'(-2) = \frac{1}{f'(0)} = \frac{1}{3 + 2\sin(0)} = \frac{1}{3}.$$