Lecture 18: The Fundamental Theorem of Calculus

18.1 The fundamental theorem of differential calculus

Suppose f is continuous on $[a, b]$ and define

$$g(x) = \int_a^x f(t)dt.$$

for all x in $[a, b]$. Then, for any x in (a, b) and $h > 0$,

$$\begin{aligned}
\frac{g(x+h) - g(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\
&= \frac{1}{h}\left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\
&= \frac{1}{h}\int_x^{x+h} f(t)dt.
\end{aligned}$$

Let M be the maximum value of f on $[x, x+h]$ and let m be the minimum value of f on $[x, x+h]$. Then

$$mh \le \int_x^{x+h} f(t)dt \le Mh,$$

so

$$m \le \frac{1}{h}\int_x^{x+h} f(t)dt \le M.$$

That is,

$$m \le \frac{g(x+h) - g(x)}{h} \le M.$$

But $\lim\limits_{h \to 0^+} m = f(x) = \lim\limits_{h \to 0^+} M$, so

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = f(x).$$

Similarly,

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = f(x).$$

Hence we have $g'(x) = f(x)$.

Fundamental Theorem of Differential Calculus If f is continuous on $[a, b]$ and

$$g(x) = \int_a^x f(t)dt,$$

then g is continuous on $[a, b]$, g is differentiable on (a, b), and $g'(x) = f(x)$ for all x in (a, b).

In other words, under the conditions of the theorem,

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

Example If

$$g(x) = \int_1^x \sqrt{1+t^2}dt,$$

then

$$g'(x) = \sqrt{1+x^2}.$$

Example If

$$f(x) = \int_4^{x^2} \sin(2t)dt,$$

then, using the chain rule,

$$f'(x) = 2x\sin(2x^2).$$

Example If

$$f(x) = \int_{4x}^5 \cos(t^2)dt,$$

then

$$f'(x) = \frac{d}{dx}\int_{4x}^5 \cos(t^2)dt = -\frac{d}{dx}\int_5^{4x} \cos(t^2)dt = -4\cos(16x^2).$$

Example If

$$g(t) = \int_t^{3t^2} \sin(4u^2)du,$$

then

$$g(t) = \int_t^0 \sin(4u^2)du + \int_0^{3t^2} \sin(4u^2)du = -\int_0^t \sin(4u^2)du + \int_0^{3t^2} \sin(4u^2)du.$$

Hence

$$g'(t) = -\sin(4t^2) + 6t\sin(36t^4).$$

18.2 The fundamental theorem of integral calculus

Now suppose f is continuous on $[a, b]$ and F is an antiderivative of f on (a, b) which is continuous on $[a, b]$. Let

$$g(x) = \int_a^x f(t)dt.$$

Then g is also an antiderivative of f, so there exists a constant c such that

$$F(x) = g(x) + c$$

for all x in $[a, b]$. Now $g(a) = 0$, so

$$F(a) = g(a) + c = c.$$

Thus

$$F(x) = g(x) + F(a),$$

which implies that

$$g(x) = F(x) - F(a).$$

In particular,

$$\int_a^b f(t)dt = g(b) = F(b) - F(a).$$

Fundamental Theorem of Integral Calculus If f is continuous on $[a, b]$ and F is an antiderivative of f which is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

We will use the notation

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example $\displaystyle\int_0^2 x^2 dx = \frac{1}{3}x^3\Big|_0^2 = \frac{8}{3}$

Example $\displaystyle\int_{-1}^3 (x^2+1)dx = \left(\frac{x^3}{3} + x\right)\Big|_{-1}^3 = (9+3) - (-\frac{1}{3} - 1) = \frac{40}{3}$

Example $\displaystyle\int_0^{\pi} \sin(x)dx = -\cos(x)\Big|_0^{\pi} = -(-1) - (-1) = 2$

Example $\displaystyle\int_0^{2\pi} \cos(x)dx = \sin(x)\Big|_0^{2\pi} = 0 - 0 = 0$

Example $\displaystyle\int_1^{9} \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_1^{9} = \frac{54}{3} - \frac{2}{3} = \frac{52}{3}$