Lecture 10: Concavity

10.1 Concave upward and concave downward

Example Note that both $f(x) = x^2$ and $g(x) = \sqrt{x}$ are increasing on the interval $[0, \infty)$, but their graphs look significantly different. This is explained by the fact that $f'(x) = 2x$, and so is an increasing function on $[0, \infty)$, whereas $g'(x) = \frac{1}{2\sqrt{x}}$, and so is a decreasing function on $(0, \infty)$.

Definition We say the graph of a function f is *concave upward* on an interval (a, b) if f' is increasing on (a, b). We say the graph of f is *concave downward* on (a, b) if f' is decreasing on (a, b).

Theorem If f is twice differentiable on an interval (a, b), then

1. $f''(x) > 0$ for all x in (a, b) implies the graph of f is concave upward on (a, b),
2. $f''(x) < 0$ for all x in (a, b) implies the graph of f is concave downward on (a, b).

10.2 Examples

Example Let $f(x) = x^2$. Then $f''(x) = 2$, so $f''(x) > 0$ for all x in $(-\infty, \infty)$. Thus the graph of f is concave upward on $(-\infty, \infty)$.

Example Let $f(x) = \sqrt{x}$. Then $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, so $f''(x) < 0$ for all x in $(0, \infty)$. Thus the graph of f is concave downward on $(0, \infty)$.

Example Let $f(x) = -x^2$. Then $f''(x) = -2$, so $f''(x) < 0$ for all x in $(-\infty, \infty)$. Thus the graph of f is concave downward on $(-\infty, \infty)$.

Example Let $f(x) = x^{\frac{2}{3}}$. Then $f''(x) = -\frac{2}{9}x^{-\frac{4}{3}}$, so $f''(x) < 0$ for all x in $(-\infty, 0)$ and in $(0, \infty)$. Thus the graph of f is concave downward on $(-\infty, 0)$ and on $(0, \infty)$.

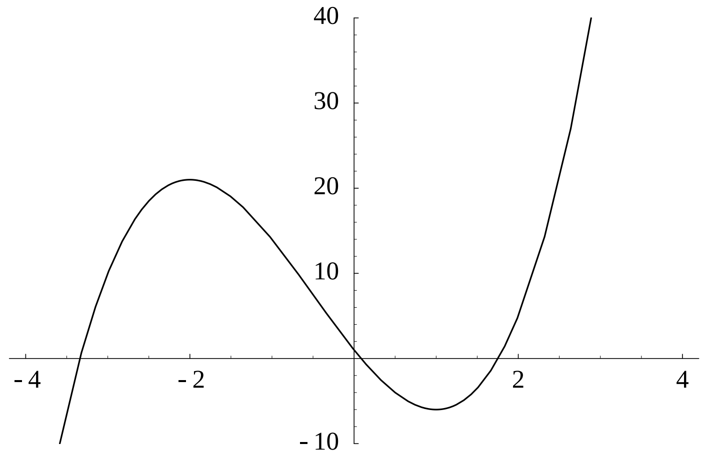
Example Let $f(x) = x^3$. Then $f''(x) = 6x$, so $f''(x) < 0$ for all x in $(-\infty, 0)$ and $f''(x) > 0$ for all x in $(0, \infty)$. Thus the graph of f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$. We call the point $(0, 0)$ where the concavity of the graph changes a *point of inflection.*

Definition A point on the graph of a function at which the concavity changes is called a *point of inflection.*

Example Let $f(x) = 2x^3 + 3x^2 - 12x + 1$. Then $f''(x) = 12x + 6$, so $f''(x) = 0$ when $x = -\frac{1}{2}$. Moreover, $f''(x) < 0$ when x is in $(-\infty, -\frac{1}{2})$ and $f''(x) > 0$ when x is in $(-\frac{1}{2}, \infty)$.

Hence the graph of f is concave upward on $(-\frac{1}{2}, \infty)$ and concave downward on $(-\infty, -\frac{1}{2})$. Moreover, $(-\frac{1}{2}, \frac{15}{2})$ is a point of inflection.

Combining this new information with our previous information on this function (namely, $f(-3) = 10$, $f(-2) = 21$, $f(1) = -6$, $f(2) = 5$, f is increasing on $(-\infty, -2)$ and on $(1, \infty)$, and f is decreasing on $(-2, 1)$), we can sketch the graph of f.



Graph of $f(x) = 2x^3 + 3x^2 - 12x + 1$

10.3 The second derivative test

Note that if $f'(c) = 0$ and $f''(x) < 0$ on an open interval (a, b) containing c, then f' is decreasing on (a, b), and hence $f'(x) > 0$ for x in (a, c) and $f'(x) < 0$ for x in (c, b). Thus f is increasing on (a, c) and decreasing on (c, b), and so f has a local maximum at c.

Second Derivative Test Suppose f'' is continuous on an open interval that contains the point c and $f'(c) = 0$. Then

1. $f''(c) < 0$ implies f has a local maximum at c,
2. $f''(c) > 0$ implies f has a local minimum at c.

10.4 More examples

Example Let $f(x) = 3x^5 - 5x^3 + 1$. Then

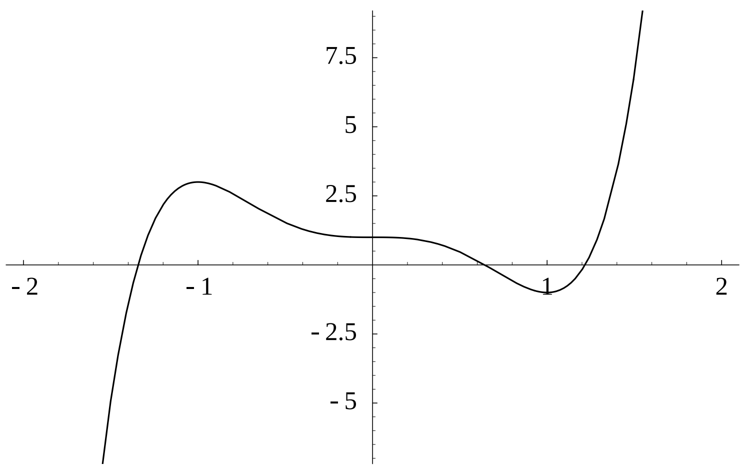
$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1).$$

Hence $f'(x) = 0$ when $x = -1$, $x = 0$, or $x = 1$. Now $f''(x) = 60x^3 - 30x$, so $f''(-1) = -30 < 0$, $f''(0) = 0$, and $f''(1) = 30 > 0$. Hence f has a local maximum of 3 at $x = -1$ and a local minimum of -1 at $x = 1$.

Moreover, we may conclude that f is increasing on $(-\infty, -1)$ and on $(1, \infty)$, and f is decreasing on $(-1, 0)$ and on $(0, 1)$. It follows that f has neither a local maximum nor a local minimum at $x = 0$.

Since $f''(x) = 60x^3 - 30x = 60x(x^2 - \frac{1}{2})$, we see that $f''(x) = 0$ when $x = -\frac{1}{\sqrt{2}}$, $x = 0$, or $x = \frac{1}{\sqrt{2}}$. Moreover, we see that $f''(x) < 0$ when x is in $(-\infty, -\frac{1}{\sqrt{2}})$ or $(0, \frac{1}{\sqrt{2}})$, and $f''(x) > 0$ when x is in $(-\frac{1}{\sqrt{2}}, 0)$ or $(\frac{1}{\sqrt{2}}, \infty)$. Hence, the graph of f is concave upward on $(-\frac{1}{\sqrt{2}}, 0)$ and $(\frac{1}{\sqrt{2}}, \infty)$, and concave downward on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(0, \frac{1}{\sqrt{2}})$. Thus $(-\frac{1}{\sqrt{2}}, \frac{7}{4\sqrt{2}} + 1)$, $(0, 1)$, and $(\frac{1}{\sqrt{2}}, -\frac{7}{4\sqrt{2}} + 1)$ are all points of inflection.

Combining this information with the values $f(-2) = -55$, $f(-1) = 3$, $f(-\frac{1}{\sqrt{2}}) = \frac{7}{4\sqrt{2}} + 1 \approx 2.2374$, $f(0) = 1$, $f(\frac{1}{\sqrt{2}}) = -\frac{7}{4\sqrt{2}} + 1 \approx -0.2374$, $f(1) = -1$, and $f(2) = 57$, we may sketch the graph of f.



Graph of $f(x) = 3x^5 - 5x^3 + 1$

Example Let $g(x) = \dfrac{x}{x^2 + 1}$. Then

$$g'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$

and

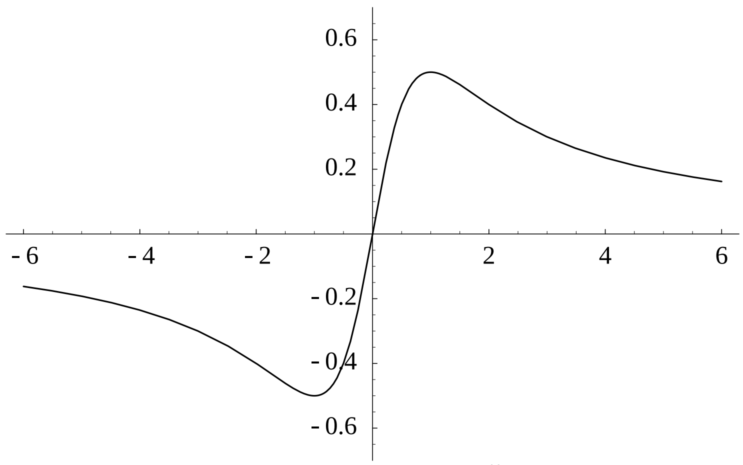
$$g''(x) = \frac{2x^3 - 6x}{(x^2 + 1)^3}.$$

Then $g'(x) = 0$ when $x = -1$ or $x = 1$, and $g''(-1) = \frac{1}{2}$ and $g''(1) = -\frac{1}{2}$. Hence g has a local maximum of $\frac{1}{2}$ at $x = 1$ and a local minimum of $-\frac{1}{2}$ at $x = -1$. Moreover, g is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.

Now $g''(x) = 0$ when $2x^3 - 6x = 2x(x^2 - 3) = 0$, that is, when $x = -\sqrt{3}$, $x = 0$, or $x = \sqrt{3}$. Also, $g''(x) < 0$ when x is in $(-\infty, -\sqrt{3})$ or $(0, \sqrt{3})$, and $g''(x) > 0$ when x is in $(-\sqrt{3}, 0)$

or $(\sqrt{3}, \infty)$. Hence the graph of g is concave downward on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$, and the graph of g is concave upward on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$. Thus there are three points of inflection: $(-\sqrt{3}, -\frac{\sqrt{3}}{4})$, $(0, 0)$, and $(\sqrt{3}, \frac{\sqrt{3}}{4})$

Adding to the above information the values $g(-2) = -\frac{2}{5}$ and $g(2) = \frac{2}{5}$ helps us sketch the graph of g, but we really need more information about the behavior of the function for values of x approaching $-\infty$ and ∞ before we can fully understand the shape of the graph of g.



Graph of $g(x) = \dfrac{x}{x^2 + 1}$