Increasing and Decreasing Functions

Mathematics 11: Lecture 24

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Definitions

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- Example: Let $f(x) = x^2$.
    - If $0 < x_1 < x_2$, then $x_1^2 < x_2^2$, so f is increasing on $(0, \infty)$.
    - If $x_1 < x_2 < 0$, then $x_1^2 > x_2^2$, so f is decreasing on $(-\infty, 0)$.

Connection with derivatives

- Suppose f is differentiable on (a, b) with $f'(x) > 0$ for all x in (a, b), and let x_1 and x_2 be points in (a, b) with $x_1 < x_2$.

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- Similarly, if $f'(x) < 0$ for all x in (a, b), it follows that f is decreasing on (a, b).

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  - if $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b).

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- When $x < -2$, both $x + 2 < 0$ and $x - 1 < 0$, so $f'(x) > 0$.
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- When $x < -2$, both $x + 2 < 0$ and $x - 1 < 0$, so $f'(x) > 0$.
- When $-2 < x < 1$, $x + 2 > 0$ but $x - 1 < 0$, so $f'(x) < 0$.
- When $x > 1$, both $x + 2 > 0$ and $x - 1 > 0$, so $f'(x) > 0$.

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- Hence we conclude that
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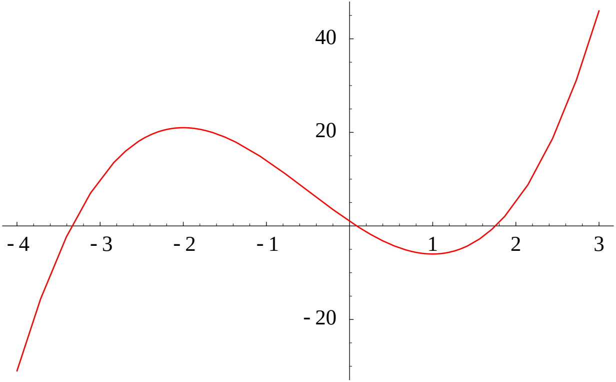
  - $f'(0) = -12$, so $f'(x) < 0$ on $(-2, 1)$,
  - $f'(2) = 24$, so $f'(x) > 0$ on $(1, \infty)$.
- Hence we conclude that
  - f is increasing on $(-\infty, -2)$,
  - f is decreasing on $(-2, 1)$,
  - and f is increasing on $(1, \infty)$.

Example (cont'd)

- Graph of $f(x) = 2x^3 + 3x^2 - 12x + 1$:



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$$f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3).$$
- Hence $f'(x) = 0$ when $x = -\sqrt{\frac{3}{5}}$, $x = 0$, or $x = \sqrt{\frac{3}{5}}$.
- Note: $x^2 > 0$ for all $x \neq 0$, $5x^2 - 3 < 0$ for
$$-\sqrt{\frac{3}{5}} < x < \sqrt{\frac{3}{5}},$$
and $5x^2 - 3 > 0$ for all other x.

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  - and increasing on $\left(\sqrt{\frac{3}{5}}, \infty\right)$.

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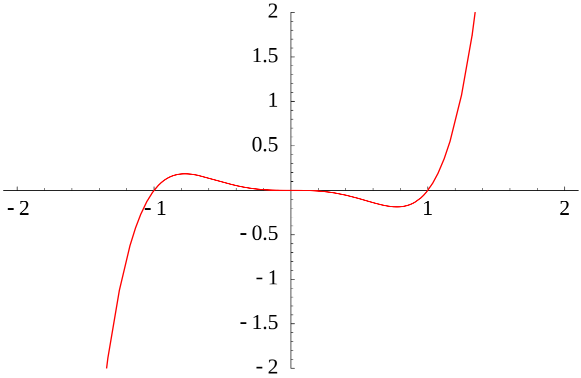

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- Moreover, $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.
- So f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

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