

Linear Approximations

Mathematics 11: Lecture 20

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- ▶ We have seen this previously with our discussion of $\sin(x) \approx x$ for x close to 0.
- ▶ That is, the linearization of $f(x) = \sin(x)$ at $x = 0$ is $L(x) = x$.

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- ▶ In fact, L is the only linear function that has this property for a given function f and point a .

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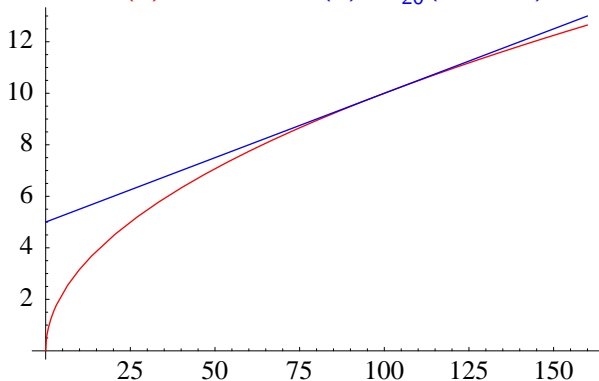
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- ▶ Note: to four decimal places, $\sqrt{101} = 10.0499$ and $\sqrt{98} = 9.8995$, so this simple approximation is very close.

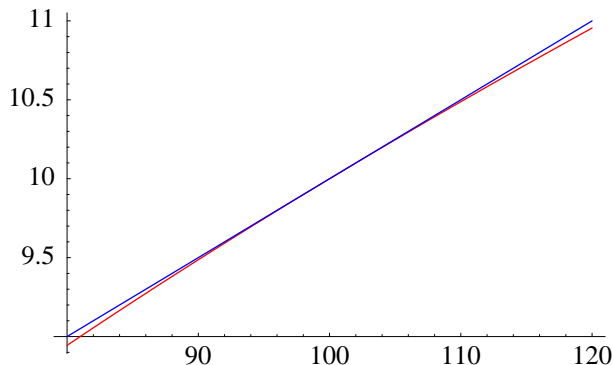
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- ▶ To four decimal places, $\sqrt[5]{1.05} = 1.0098$.