Derivatives
Mathematics 11: Lecture 14

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Definition

- The *derivative* of a function f is the function f' with value at x given by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$
provided the limit exists.

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- We say f is *differentiable* on an open interval (a, b) if $f'(x)$ exists for all x in (a, b).

Example

- If $f(x) = x^2$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

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- For example, $f'(1) = 2$, as we have seen before.

Example

- If $f(x) = \sqrt{x}$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Example (cont'd)

- For example, $f'(9) = \frac{1}{6}$, so the equation of the line tangent to the graph of $y = \sqrt{x}$ at $(9, 3)$ is

$$y = \frac{1}{6}(x - 9) + 3.$$

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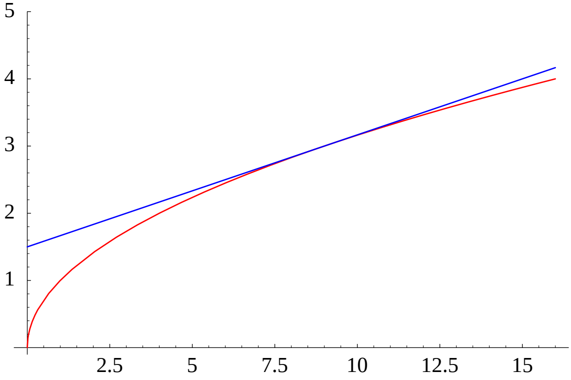
- Note: the domain of f is $[0, \infty)$, while the domain of f' is $(0, \infty)$.

Example (cont'd)

- Graphs of $y = \sqrt{x}$ and $y = \frac{1}{6}(x - 9) + 3$:



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- Then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(m(x+h)+b) - (mx+b)}{h} \\
&= \lim_{h \to 0} \frac{mh}{h} \\
&= \lim_{h \to 0} m \\
&= m.
\end{aligned}
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- That is, the derivative at any point is the slope of the line $y = mx + b$, which is the graph of f.

Slopes

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- That is, we may think of $f'(x)$ as the slope of the graph of f at x.

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- Now
$$\frac{|h|}{h} = \begin{cases} -1, & \text{if } h < 0, \\ 1, & \text{if } h > 0. \end{cases}$$
- Hence
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$
and
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1.$$

Example (cont'd)

- Thus

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  does not exist.
- In other words, f is not differentiable at $x = 0$.
- Note: f is continuous at 0.
- Hence a function may be continuous at a point without being differentiable at that point.

Theorem

- If f is differentiable at a, then f is continuous at a.

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- Equivalently, we need to show $\lim_{h \to 0} (f(a+h) - f(a)) = 0$.
- Now

$$\begin{aligned} \lim_{h \to 0} (f(a+h) - f(a)) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \times h \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \lim_{h \to 0} h \\ &= f'(a) \times 0 = 0. \end{aligned}$$

Higher-order derivatives

- Note: if a function f is differentiable on an interval (a, b), then its derivative f' is also function on (a, b). Hence we can ask if f' is itself differentiable on (a, b).

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- If f' is differentiable, we denote its derivative by f'', the *second derivative* of f.
- Example
  - If $f(x) = x^2$, then we know $f'(x) = 2x$.
  - Hence, from our work above, we know $f''(x) = 2$.

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- Example
  - If $f(x) = x^2$, then, as above, $f'(x) = 2x$ and $f''(x) = 2$.
  - Then $f'''(x) = 0$ and $f^{(4)}(x) = 0$.

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- In general, the nth derivative is denoted $\frac{d^ny}{dx^n}$.
- Example
  - If $y = x^2$, then

$$\frac{dy}{dx} = 2x, \frac{d^2y}{dx^2} = 2, \frac{d^3y}{dx^3} = 0, \text{ and } \frac{d^4y}{dx^4} = 0.$$

Acceleration

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- Then $a = \frac{dv}{dt} = f''(t)$ is the *acceleration* of the object at time t.