Vertical Asymptotes

Mathematics 11: Lecture 10

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Unbounded limits and asymptotes

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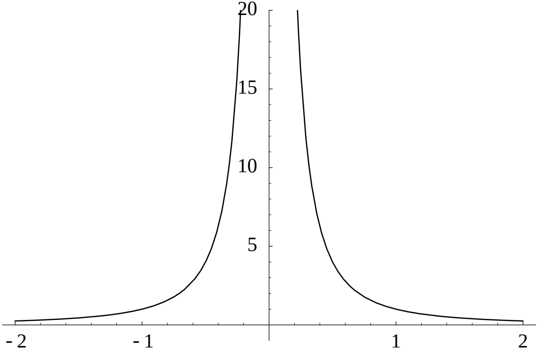
- This notation means:
  - the limit does not exist, and
  - the reason it does not exist is that the $f(x)$ increases without bound as x approaches 0.

Asymptotes

- Geometrically, in the previous example the vertical line $x = 0$ is a vertical asymptote for the graph of f:



Graph of $f(x) = \dfrac{1}{x^2}$

Definition

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- Similarly, we write $\lim_{x \to a} f(x) = -\infty$ if for every negative number K there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < K$.
- Note: we can define such notation for one-side limits as well.

Definition

- We say the line $x = a$ is a *vertical asymptote* for the graph of a function f if

$$\lim_{x \to a^-} f(x) = -\infty,$$

$$\lim_{x \to a^-} f(x) = \infty,$$

$$\lim_{x \to a^+} f(x) = -\infty,$$

or

$$\lim_{x \to a^+} f(x) = \infty.$$

Example

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- So we write

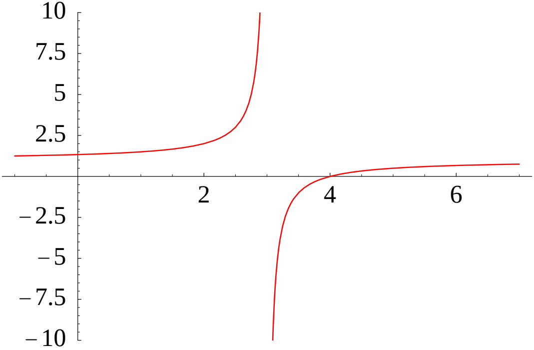
$$\lim_{x\to 3^-} \frac{4-x}{3-x} = \infty$$

and

$$\lim_{x\to 3^+} \frac{4-x}{3-x} = -\infty.$$

Example (cont'd)

- Hence the vertical line $x = 3$ is a vertical asymptote for the graph of f, as seen in the graph below.



Graph of $f(x) = \dfrac{4 - x}{3 - x}$

Example

- Note that

$$\frac{x^2+x-6}{x^2-4x+4} = \frac{(x-2)(x+3)}{(x-2)(x-2)} = \frac{x+3}{x-2} \text{ for } x \neq 2.$$

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- Hence
$$\lim_{x\to 2^-} \frac{x^2+x-6}{x^2-4x+4} = \lim_{x\to 2^-} \frac{x+3}{x-2} = -\infty$$
and
$$\lim_{x\to 2^+} \frac{x^2+x-6}{x^2-4x+4} = \lim_{x\to 2^+} \frac{x+3}{x-2} = \infty$$

Example

- $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin(x)}{\cos(x)} = \infty$

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