Senior Exam

Furman University
Wylie Mathematics Tournament

March 4, 2000

1. Suppose x and y are the digits for which $x01y$ is divisible by both 8 and 9. What is the sum of x and y?

(1) 5 (2) 6
(3) 7 (4) 8
(5) None of the above

2. Given that $\log_b(a^2) = 3$, what is $\log_a(b^2)$?

(1) $\frac{4}{9}$ (2) $\frac{4}{3}$
(3) $\frac{3}{4}$ (4) $\frac{3}{9}$
(5) None of the above

3. When Darby remarked that the apples seemed very small this week, the seller offered to throw an extra apple into the dollar basket. Darby noted that this reduced the price *per dozen* by 5 cents and she then purchased the basket. How many apples did she get for her dollar?

(1) 15 (2) 16
(3) 17 (4) 18
(5) None of the above

4. The natives of the Isle of Kram Dradoow use a base other than 10 for enumeration. A visitor to Kram Dradoow saw a teacher write the problem

$$1.\quad \frac{2}{3} \times \boxed{} = 13$$

$$2.\quad \frac{3}{4} \times 22 = \boxed{}$$

on the blackboard. When a student correctly filled in 21 for the answer to the first problem, the visitor was able to deduce the answer to the second problem. What answer should go in the second box? Be sure to write your answer in the Kram Dradoowian system.

(1) 14 (2) 15
(3) 16 (4) 17
(5) None of the above

5. In the following long division, the digits in the places marked with a $\star$ have been smudged.

$$\begin{array}{cccccccc}
 & & & & & \star & \star & \star \\ \cline{5-8}
\star & \star & \sqrt{} & & \star & \star & \star & \star \\
 & & & & 3 & 9 & & \\ \cline{5-6}
 & & & & & \star & \star & \\
 & & & & & 5 & \star & \\ \cline{6-7}
 & & & & & \star & \star & 4 \\
 & & & & & \star & \star & \star \\ \cline{6-8}
 & & & & & & & 0
\end{array}$$

What was the quotient?

(1) 248 (2) 328
(3) 342 (4) 348
(5) None of the above

6. Express the sum of the repeating decimals

$$.68686868\ldots + .0777777777777\ldots$$

as a repeating decimal.

(1) $.765656565656\ldots$ (2) $.76666666666\ldots$
(3) $.767676767676\ldots$ (4) $.76464646\ldots$
(5) None of the above

7. Augustus DeMorgan, a nineteenth century mathematician, once remarked "I was x years old in the year x^2." In what year was DeMorgan born?

(1) 1803 (2) 1804
(3) 1805 (4) 1806
(5) None of the above

8. Find the largest x so that $|x| < \frac{1}{2}$, and

$$x^4 + x^3 - 3x^2 + x = 0.$$

(1) 0 (2) $\frac{\sqrt{2}-1}{4}$
(3) $\frac{\sqrt{2}-1}{2}$ (4) 1
(5) None of the above

9. How many terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

must be added for the sum to first exceed .999?

(1) 10 (2) 11

(3) 12 (4) 13

(5) None of the above

10. In a survey of 100 persons it was found that 39 subscribed to TV Guide, 26 subscribed to Time and 6 subscribed to Scientific American. A total of 15 subscribed to *at least two* of these magazines and 2 subscribed to *all three*. How many persons did not subscribe to any of the three?

(1) 46 (2) 47

(3) 48 (4) 49

(5) None of the above

11. Suppose that

$$\log_{10}(x-2) + \log_{10} y = 0$$

and

$$\sqrt{x} + \sqrt{y-2} = \sqrt{x+y}.$$

What is $x + y$?

(1) 2 (2) $2\sqrt{2}$

(3) $2 + 2\sqrt{2}$ (4) $4 + 2\sqrt{2}$

(5) None of the above

12. A 360 foot long passenger train completely passes a 1400 foot freight train traveling in the same direction in 60 seconds. When moving in opposite directions the trains pass in 12 seconds. (The passing time is the total period during which any part of one train is along side a part of the other.) What is the speed of the passenger train? (In feet per second.)

(1) 77 (2) 88

(3) 99 (4) 104

(5) None of the above

13. The nth triangular number is defined to be the sum of the first n positive integers. For example, the 4th triangular number is $1 + 2 + 3 + 4 = 10$. In the first 100 terms of the sequence

$$1, 3, 6, 10, 15, 21, 28, \dots$$

of triangular numbers, how many are divisible by 7?

(1) 24 (2) 25

(3) 26 (4) 27

(5) None of the above

14. Suppose the positive integers are written in succession

$$12345678910111213141516\dots.$$

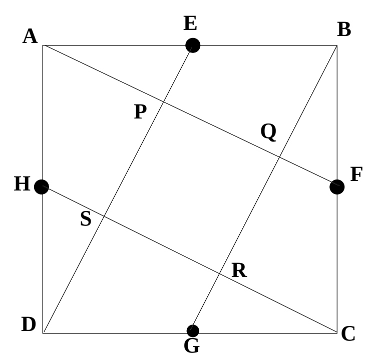
What digit appears in the thousandth place?

(1) 0 (2) 1

(3) 2 (4) 3

(5) None of the above

15. In the figure $ABCD$ is a square of side 2 and E, F, G, and H are the midpoints of the sides. What is the area of square $PQRS$?



(1) $\frac{3}{4}$ (2) $\frac{4}{5}$

(3) $\frac{5}{6}$ (4) $\frac{6}{7}$

(5) None of the above

16. A radiator holds 16 liters of an antifreeze-water mixture that is 30% antifreeze. Your job is to drain just the right amount of fluid from the radiator so that when the radiator is refilled with pure antifreeze, the mixture becomes 50% antifreeze. How many liters do you need to drain?

(1) $\frac{29}{7}$ (2) $\frac{30}{7}$

(3) $\frac{31}{7}$ (4) $\frac{32}{7}$

(5) None of the above

17. You are being evaluated in three different categories. In category I you can receive either a 0, 1, or 2. In categories II and III you can receive either a 0, 1, 2, 3, or 4. Your ultimate rating is the sum of the points you receive in each category. In how many different ways could you end up with a rating of 5?

(1) 10 (2) 11

(3) 12 (4) 13

(5) None of the above

18. One of the most prolific mathematicians in history was Leonhard Euler (1707 – 1783). One of his many discoveries was that the sum of the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

is $\frac{\pi^2}{6}$. What is the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots ?$$

(1) $\frac{\pi^2}{7}$ (2) $\frac{\pi^2}{8}$

(3) $\frac{\pi^2}{9}$ (4) $\frac{\pi^2}{10}$

(5) None of the above

19. Find a rational number p such that

$$2^p \cdot 4^p \cdot 8^p \cdots (2^{100})^p = 2^{100}.$$

(1) $\frac{2}{101}$ (2) $\frac{1}{50}$

(3) $\frac{2}{103}$ (4) $\frac{1}{51}$

(5) None of the above

20. Let n be the number of sequences of integers a_1, a_2, a_3, a_4, a_5 that exist where

$$0 < a_1 < a_2 < a_3 < a_4 < a_5 < 100.$$

What is n congruent to modulo 10?

(1) 2 (2) 4

(3) 6 (4) 8

(5) None of the above

21. Determine the value of the infinite product

$$\sqrt{5} \cdot \sqrt{\sqrt{5}} \cdot \sqrt{\sqrt{\sqrt{5}}} \cdot \sqrt{\sqrt{\sqrt{\sqrt{5}}}} \cdots .$$

(1) $\sqrt{5}$ (2) $5\sqrt{5}$

(3) 5 (4) $\sqrt{10}$

(5) None of the above

22. Given that z satisfies $z + \frac{1}{z} = 2\cos 13°$, find an angle B so that $0 < B < \frac{\pi}{2}$ and $z^2 + \frac{1}{z^2} = 2\cos(B)$.

(1) 23° (2) 24°

(3) 25° (4) 26°

(5) None of the above

23. If three students eat 6 bagels in 9 minutes, how long does 1 student need to eat 2 bagels?

(1) 8 minutes (2) 9 minutes

(3) 10 minutes (4) 11 minutes

(5) None of the above

24. If x and y are two positive integers whose product is 1,000,000 and neither of which has any zero digits, what is $x + y$?

(1) 15,432 (2) 44,431

(3) 14,689 (4) 13,325

(5) None of the above

25. At the thrift store, Hannah got a real bargain – a programmable scientific calculator that never overflows. The only catch is that the display only shows the one's digit. If Hannah put in a 7 and pressed the square button 2000 times, what did the display read at the end?

(1) 1 (2) 9

(3) 3 (4) 7

(5) None of the above

26. How many positive integers less than 1000 have a 1 in their base 16 expansion?

(1) 348 (2) 349

(3) 350 (4) 351

(5) None of the above

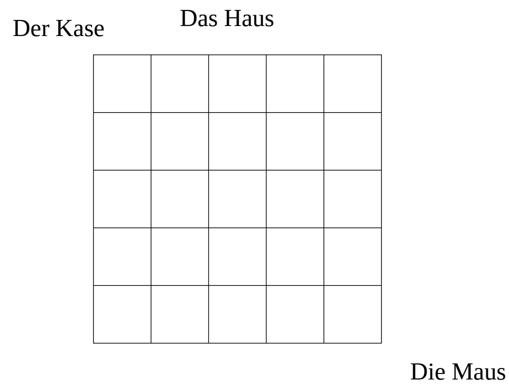
27. Madison's dog chewed on her homework before she could finish it. The fragment saved from the horrible canine's mouth reveal only the two terms of highest degree of the polynomial $p(x)$. It looked like $p(x) = x^{18} - 3x^{17} + \ldots$. Madison found roots $1, 2, 3, \ldots 17$. What is the missing root?

(1) 18 (2) 17

(3) -18 (4) -151

(5) None of the above

28. A hungry blind Maus in a German Haus stands at the lower right corner of a grid of beams and smells a nice piece of käse (=cheese) at the opposite corner. If the mouse can only move north and west, how many possible paths are there to the cheese?

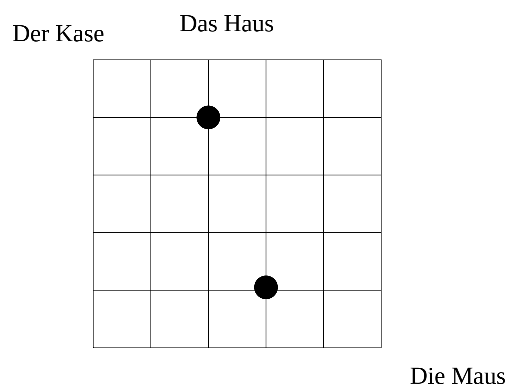


(1) 248 (2) 250

(3) 252 (4) 254

(5) None of the above

29. The owners of the Haus hid two traps on the grid as shown. Assume that the mouse heads for the cheese as in the previous problem, and let p be the probability that he won't get caught. Then p is closest to:



(1) .3 (2) .35

(3) .4 (4) .45

(5) None of the above

30. Consider the following sequence of numbers $x_0, x_1, x_2, \ldots$ defined by setting $x_0 = 100$, and thereafter setting $x_{n+1} = \dfrac{x_n + 10}{2x_n}$. Thus, for example, $x_1 = .55$. Estimate the value of $x_{100000000}$ correctly to within 10 decimal places.

(1) 2.1 (2) 2.4

(3) 2.7 (4) 3.0

(5) None of the above

31. A famous cubic from history is the equation $x^3 = 15x + 4$. All three roots of this equation are real numbers. One of the roots r can be expressed via the formula

$$r = \sqrt[3]{2 + i\sqrt{121}} + \sqrt[3]{2 - i\sqrt{121}}$$

where i is the "imaginary" number whose square is -1. Find a simpler expression for r.

(1) $\frac{2\pi}{3}$ (2) $\frac{37}{7}$

(3) 3 (4) 4

(5) None of the above

32. A 6-digit number has its first digit a 9. If you move it to the last digit instead, you get a number which is only one fourth the size of the original number. What is the sum of the digits of the original number?

(1) 27 (2) 28

(3) 29 (4) 30

(5) None of the above

Bonus Questions: Show all your work. The solution to number 33 should be written on the green sheet labeled "41", and the solution to number 34 should be written on the red sheet labeled "42."

33. An 8×11 piece of paper is folded by bringing together opposite vertices. Find the length of the crease.

34. The sum of a certain number of positive integers is 20. Find, with proof, the largest that their product can be.