1. The polynomial

$$p(x) = 3x^3 - 5x^2 - 12x + 20$$

has three distinct roots $x_1 < x_2 < x_3$. When x_2 is represented as a decimal, what is its tenths digit?

(a) 0 (b) 3
(c) 6 (d) 9
(e) None of the above

2. What is the imaginary part of the complex number

$$\frac{-4+7i}{1+2i}?$$

(a) $-1/2$ (b) 2
(c) 3 (d) $7/2$
(e) None of the above

3. You have a rectangular sheet of paper measuring 13 inches by 8 inches. Using a pair of scissors and making a single cut along a straight line across the rectangle, remove the largest possible square from the rectangle. Set the square aside and repeat this procedure on the residual piece. Continue this process until you have nothing but a pile of squares. How many squares do you have in your pile?

(a) 5 (b) 6
(c) 7 (d) 8
(e) None of the above

4. A parabola $p(x) = ax^2 + bx + c$ with vertex at $V(3,3)$ intersects the parabola $y = x^2$ at a single point. What is a?

(a) -1 (b) $-3/4$
(c) $3/4$ (d) $-1/2$
(e) None of the above

5. Let $b > 0$ be an integer and suppose that $(237)_b = (159)_{10}$. What is $(345)_b$ when written in base 10?

(a) 229 (b) 230
(c) 231 (d) 232
(e) None of the above

6. A certain bicycle has two different gears at the pedal: one has a diameter of 8 inches, while the other has a diameter of 10 inches. At the back wheel there are two gears, measuring 2 and 3 inches in diameter. Assuming that you pedal at the same rate, which of the following is closest to the percent increase in speed in shifting from the 8 : 3 linking to the 10 : 2 linking?

(a) 66 (b) 78
(c) 80 (d) 87
(e) None of the above

7. In Figure 1, three squares are joined to form a triangle. If the squares measure 4, 13 and 15 units on the side, what is the area of this entire figure?

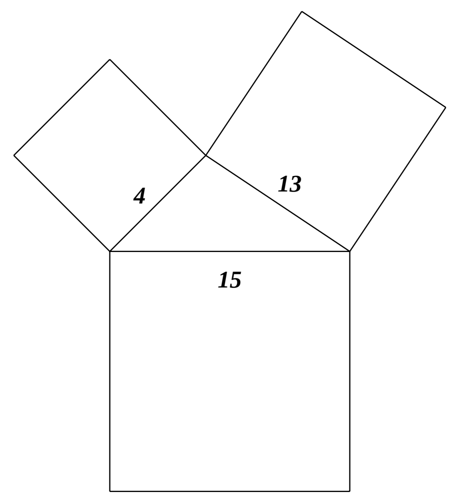

Fig. 1

(a) 434 (b) 440
(c) 476 (d) 504
(e) None of the above

8. If $\log_3(x) = p$ and $\log_7(x) = q$, which of the following yields $\log_{21}(x)$?

(a) pq
(b) $1/(p+q)$
(c) $1/(p^{-1}+q^{-1})$
(d) $pq/(p^{-1}+q^{-1})$
(e) None of the above

9. Let A denote the set of integers between 1 and 1000 which are divisible by 12. Let B denote the set of integers between 1 and 1000 which are divisible by 18. How many elements are in the set $A \cup B$?

(a) 108 (b) 109

(c) 110 (d) 111

(e) None of the above

10. Consult Figure 2. How many squares of any size are there in this figure?

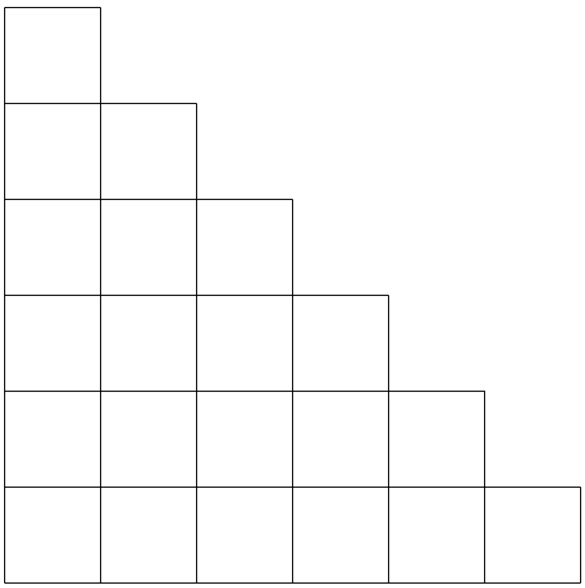

Fig. 2

(a) 30 (b) 32

(c) 34 (d) 36

(e) None of the above

11. A baby's stacking toy consists of a peg on stand and 10 rings. The rings are identical in every way except color. There are 2 red, 3 blue and 5 green rings. Assuming that all 10 rings are used, how many distinguishable stackings are there?

(a) 30 (b) 1680

(c) 2520 (d) 3630

(e) None of the above

12. Let $a_0 = 1$ and $b_0 = 2$. For all $n \geq 1$, let

$$\begin{cases} a_n = 2a_{n-1} + 3b_{n-1} \\ b_n = a_{n-1} - 2b_{n-1} \end{cases}$$

What is the remainder when a_3 is divided by 4?

(a) 0 (b) 1

(c) 2 (d) 3

(e) None of the above

13. Let $p(x) = x^2 + x - 2$ and let $q(x) = x^3 + ax^2 + bx + c$. For each real number r, we will assume that $q(r) = 0$ if and only if $p(r) = 0$. What is the largest possible value of $a - b + c$?

(a) 3 (b) 4

(c) 5 (d) 6

(e) None of the above

14. Toss a fair coin repeatedly. Each time you get a head, give yourself one point. Each time you get a tail, give yourself two points. What is the probability that at some time in the game you will have exactly 5 points?

(a) 7/32 (b) 14/32

(c) 21/32 (d) 28/32

(e) None of the above

15. Consider the following regions in the plane:

$$R_1 = \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$$
$$R_2 = \{(x, y) : x^2 + y^2 \leq 4/3\}$$

The area of the region $R_1 \cap R_2$ can be expressed as

$$\frac{a\sqrt{3} + b\pi}{9},$$

where a and b are integers. Find $a + b$.

(a) 2 (b) 3

(c) 4 (d) 5

(e) None of the above

16. A certain pyramid has a square base (with sides of length 1) and four lateral faces, each of which is an equilateral triangle (see Figure 3). A bug is situated at point A and wishes to crawl on the faces of the pyramid to point B. Assuming that point A is half way up the pyramid, what is the length of the shortest trip he can take?

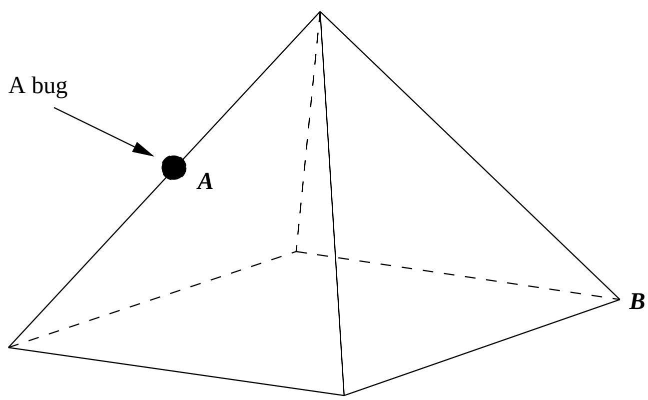

Fig. 3

(a) $\sqrt{3}$ (b) 2

(c) $\sqrt{7}/2$ (d) $\sqrt{13}/3$

(e) None of the above

17. Inscribe a regular n–gon within a circle of radius 1. What is the smallest integer n such that the perimeter of the n–gon is strictly less than the number of corners of the n–gon?

(a) 5 (b) 6

(c) 7 (d) 8

(e) None of the above

18. It is well known that

$$\cos^{-1}\left[\sin\left(\frac{23\pi}{3}\right)\right] = \frac{a\pi}{b},$$

where a and b are positive relatively prime integers. Find $a+b$.

(a) 4 (b) 5

(c) 7 (d) 11

(e) None of the above

19. Let

$$A = \{1,2,3,4,5,6,7,8,9,10\}$$
$$B = \{1,2,3\}$$

How many subsets of A contain at least one element of B?

(a) 512 (b) 625

(c) 784 (d) 896

(e) None of the above

20. Consider the triangle pictured in Figure 4. Assuming that $0 < \alpha < \frac{\pi}{2}$, how many triangles have integer values for C?

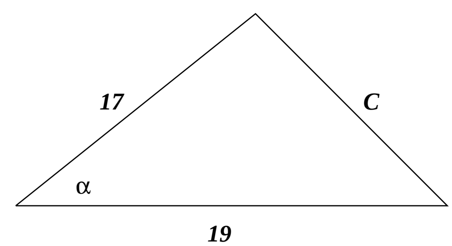

Fig. 4

(a) 22 (b) 23

(c) 24 (d) 25

(e) None of the above

21. Find the sum of the solutions of

$$2 \cdot 8^x + 32 \cdot 8^{-x} = 65.$$

(a) 2/3 (b) 4/3

(c) 2 (d) 6

(e) None of the above

22. Evaluate

$$\lfloor \sqrt[3]{1} \rfloor + \lfloor \sqrt[3]{2} \rfloor + \cdots + \lfloor \sqrt[3]{124} \rfloor.$$

(a) 401 (b) 402

(c) 403 (d) 404

(e) None of the above

23. What is the thousands digit of the coefficient of $xy^2z^3w^2$ in the expansion of

$$(x + 2y + 3z + 5w)^8?$$

(a) 3 (b) 4

(c) 5 (d) 6

(e) None of the above

24. An urn contains 10 balls colored either black or red. When selecting two balls from the urn at random, the probability that you select a ball of each color is 8/15. Assuming that the urn contains more black balls than red balls, what is the probability of obtaining at least one black ball when selecting two balls from the urn?

(a) 4/10 (b) 7/3

(c) 39/45 (d) 41/45

(e) None of the above

25. Consider the triangle pictured in Figure 5. The tangent of $\alpha/2$ can be expressed as a/b, where a and b are relatively prime positive integers. Find $a + b$.

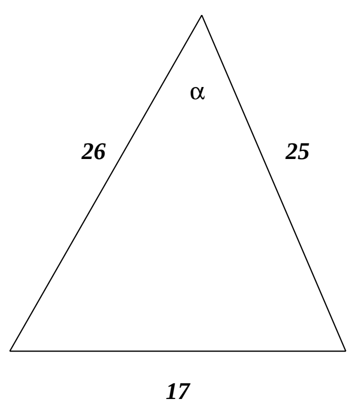

Fig. 5

(a) 17 (b) 19

(c) 21 (d) 23

(e) None of the above

26. The natural domain of the function

$$f(x) = \sin^{-1}\left(4x^2 + 6x + 1\right)$$

is a finite collection of disjoint intervals. What is the sum of the lengths of these intervals?

(a) 1/2 (b) 1

(c) 3/2 (d) 2

(e) None of the above

27. Let $0 < \theta_1 < \theta_2 < \theta_3 < \cdots$ denote the positive solutions of the equation

$$\sin(4\theta) = \cos(2\theta).$$

Naturally, $\theta_{15} = a\pi/b$, where a and b are positive relatively prime integers. What is $a + b$?

(a) 47 (b) 53

(c) 57 (d) 61

(e) None of the above

28. Two circles of equal radii are inscribed within a regular hexagon, as pictured in Figure 6. The sides of the hexagon are of length 1, and the circles are tangent at T. The common radius of these circles can be expressed as $a\sqrt{3} + b$, where a and b are integers. Find $a + b$.

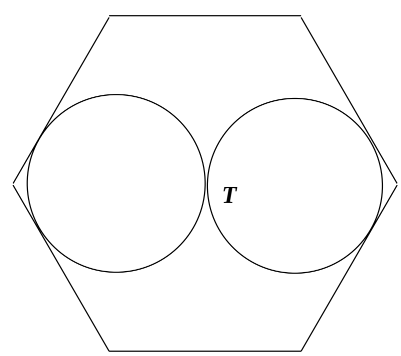

Fig. 6

(a) -2 (b) -1

(c) 1 (d) 2

(e) None of the above

29. Approximately on what percentage of the interval $[-10, 10]$ is the inequality

$$x > 3 + \frac{7}{x+3}$$

satisfied?

(a) 30 (b) 35

(c) 40 (d) 45

(e) None of the above

30. Center a sphere of radius 1 at each corner of a cube having sides of length 2. What the radius of the largest sphere that can be centered within the cube and which touches each of the remaining spheres in at most one point?

(a) $\sqrt{2} - 1$ (b) $\sqrt{3} - 1$

(c) 1 (d) $\sqrt{5} - 1$

(e) None of the above

31. A laser is situated at the point $(0, 3/4)$ in the plane and directs its beam at a slope of $2/3$ into the first quadrant. The line segments $(0, 0)$ to $(0, 1)$, $(0, 1)$ to $(1, 1)$, $(1, 1)$ to $(1, 0)$, and $(1, 0)$ to $(0, 0)$ are mirrored. How many reflections will the beam make before returning to the point $(0, 3/4)$?

(a) 7 (b) 9

(c) 11 (d) 13

(e) None of the above

32. Let $x_1 < x_2 < x_3 < x_4$ be four numbers. The four numbers can be paired in six different ways, and here are the average values of these pairings: 3, 4, 5, 7, 8, and 9. What is the largest possible value of x_4?

(a) 10 (b) 12

(c) 14 (d) 16

(e) None of the above

33. A triangle with sides of length 13, 14 and 15 inches is to be cut whole from a rectangular sheet of paper. Expressed in square inches, what is the minimum area that this rectangular sheet can have?

(a) 168 (b) 174

(c) 188 (d) 202

(e) None of the above

34. Eight circles each of radius 1 unit have their centers on a common circle with radius R. If the eight circles are to be mutually disjoint, then the set of all possible values of R is an open interval (ρ, ∞). It can be shown that $\rho^2 = a + b\sqrt{2}$, where a and b are integers. What is $a + b$?

(a) 4 (b) 5

(c) 6 (d) 7

(e) None of the above

35. The four faces of the pictured tetrahedron are equilateral triangles (see Figure 7). If $|AB| = 2$ and $|AE| = |AF| = 1$, then the area of $\triangle CEF$ can be expressed as $\sqrt{11}/a$, where a is an integer. Find a.

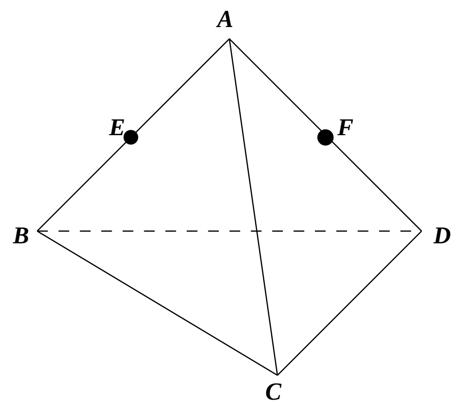

Fig. 7

(a) 2 (b) 3

(c) 4 (d) 5

(e) None of the above

36. The real number

$$\sqrt{2} + \sqrt{6 - 2\sqrt{8}}$$

is equal to which of the following numbers?

(a) 2 (b) 3

(c) $\sqrt{2} + \sqrt{3}$ (d) $2\sqrt{6}$

(e) None of the above

37. Consider Figure 8. Here is some relevant information: CA is tangent to the circle at A; O is the center of the circle; $|OA| = 1$ and $|PB| = 1/2$; OC intersects the circle at P; and PB intersects CA in a right angle. Let R denote the plane region bounded above by the arc PA, bounded below by the segment BA, and bounded on the left by the segment BP. The area of R can be expressed as $(a\sqrt{3} + b\pi)/24$, where a and b are integers. Find $a + b$.

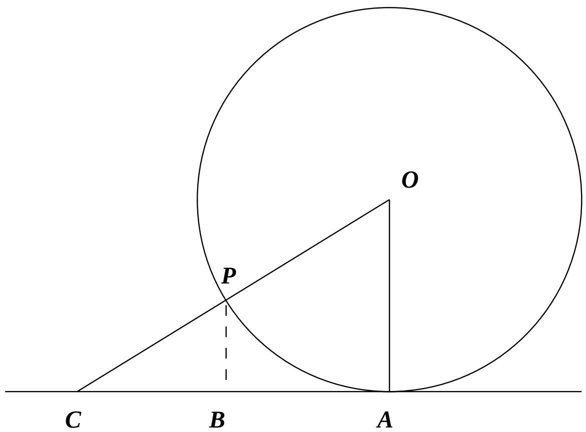

Fig. 8

(a) 5 (b) 7

(c) 9 (d) 11

(e) None of the above

38. Let $f : \{1, 2, 3, \ldots\} \to R$ satisfy the following properties:

(i) For each prime number p and each postive integer m, $f(p^m) = m + 1$.

(ii) Whenever k and l are relatively prime,

$$f(kl) = f(k)f(l).$$

Evaluate $f(6552)$.

(a) 36 (b) 40

(c) 48 (d) 56

(e) None of the above

39. Using a number 2 pencil, connect the points $(0, 0)$, $(0, 1/2)$, $(1/2, 1/2)$, $(1, 1)$, $(1, 0)$ and $(0, 0)$ in that order within the plane. The area of the resulting figure can be expressed in the form a/b, where a and b are positive relatively prime integers. What is $a + b$?

(a) 10 (b) 11

(c) 12 (d) 13

(e) None of the above

40. A triangle has sides of length 10, 17 and 21. Let α denote the angle opposite the side of length 21. Find $\lfloor \tan(\alpha) \rfloor$.

(a) -9 (b) -8

(c) -7 (d) -6

(e) None of the above

Bonus Questions

41. Show that there does *not* exist a function

$$f : (0, \infty) \to (0, \infty)$$

such that

$$f(x) + f(x^{-1}) = \frac{x^3 + 1}{x}.$$

42. Let $f_0 = 1$, $f_1 = 3$, and, for $n \geq 2$, let

$$f_n = 2f_{n-1} + 7f_{n-2}.$$

Show that $f_n \leq 4^n$ for all $n \geq 0$.