Furman University Wylie Mathematics Tournament – Junior Examination 1999

1. Let $s(n)$ denote the sum of the digits of n. Let $s^2(n)$ denote $s(s(n))$, $s^3(n) = s(s(s(n)))$ and so on. What is the value of $s^{1999}(1999)$?

(a) 28 (b) 10

(c) $\star 1$ (d) 8

(e) None of the above

2. How many of the numbers between 100 and 199 don't have exactly two identical digits?

(a) $\star 73$ (b) 74

(c) 75 (d) 76

(e) None of the above

3. An automobile radiator holds 6 liters of liquid and is full of a liquid which is 30% antifreeze. How many liters of this liquid should be replaced by pure antifreeze in order to obtain a mixture which is 50% antifreeze?

(a) 2 (b) $\star \frac{12}{7}$

(c) $\frac{4}{3}$ (d) 3

(e) None of the above

4. Suppose that the operation $*$ is defined by $\alpha * \beta = \alpha + 4\beta$. For what value of z does $\alpha * (\beta * \alpha) = (z\alpha) * \beta$?

(a) $\star 17$ (b) 19

(c) 10 (d) 4

(e) None of the above

5. In triangle ABC, the point D lies on $\overline{BC}$. If $AB = AD = CD$ and $AC = BC$, then what is the measure of $\angle ACD$ in degrees?

(a) 30 (b) 35

(c) $\star 36$ (d) 40

(e) None of the above

6. What is the sum of the positive prime factors of the number 23,595? Repeated prime factors should be repeated in the sum.

(a) $\star 43$ (b) 45

(c) 47 (d) 49

(e) None of the above

7. When simplified, an equivalent expression to $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x+y}{y}+\frac{x+y}{x}}$ is:

(a) $\frac{1}{x-y}$ (b) $x + y$

(c) $\star \frac{1}{y+x}$ (d) 1

(e) None of the above

8. If $f(x) = x^2 - 5x + 4$, then an expression equivalent to $\frac{f(x+a)-f(x)}{a}$ is

(a) 0 (b) $\frac{a^2-5a}{a}$

(c) $2x - 5$ (d) $\star 2x - 5 + a$

(e) None of the above

9. Given $f(x, y) = \frac{x^2+y^2}{x^2-y^2}$, which of the following is a true statement:

(a) $\star f(\frac{1}{x}, \frac{1}{y}) = -f(x, y)$ (b) $f(\frac{1}{x}, \frac{1}{y}) = f(x, y)$

(c) $f(\frac{1}{x}, y) = f(x, y)$ (d) $f(x, \frac{1}{y}) = f(x, y)$

(e) None of the above

10. Person A can do a piece of work in 10 days. After he has worked for 2 days, B comes and joins him and together they finish the work in 3 more days. In how many days could B have done the work by himself?

(a) $\star 6$ (b) 7

(c) 8 (d) 9

(e) None of the above

11. The horsepower of a steam engine varies jointly as the average pressure in the cylinder and the speed of rotation. When the average pressure is 400 pounds per square inch and the engine is making 750 revolutions per minute, the horsepower is 100. What is the horsepower when the average pressure is 300 pounds per square inch and the engine is making 10 revolutions per second?

(a) 6 (b) 1

(c) $\star 60$ (d) 360

(e) None of the above

12. Let $A = 2^{(3^4)}$, $B = 4^{(2^3)}$, and $C = 3^{(4^3)}$. List A, B, and C in increasing order.

(a) $A\ B\ C$ (b) $A\ C\ B$

(c) $\star B\ A\ C$ (d) $B\ C\ A$

(e) None of the above

13. Persons A and B are 30 km apart. If they leave at the same time and travel in the same direction, A overtakes B in 60 hours. If they walk toward each other they meet in 5 hours. What is the sum of their rates, measured in km per hour?

(a) $\star 6$ (b) 7

(c) 8 (d) 9

(e) None of the above

14. Two pipes together can fill a reservoir in 6 hours and 40 minutes. Find the the number of hours that the smaller pipe will take to fill the reservoir if one of the pipes can fill it in 3 hours less time than the other, working alone.

(a) $\star 15$ (b) 14

(c) 13 (d) 12

(e) None of the above

15. The equation $4x^2 - 8kx + 9 = 0$ has roots whose difference is 4. Which of the following is a possible value of k?

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) $\star\frac{5}{2}$ (d) $\frac{7}{2}$

(e) None of the above

16. Find the sum of all real solutions to

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3.$$

(a) $\star\frac{5}{3}$ (b) 0

(c) 6.5 (d) $\frac{3+\sqrt{3}}{2}$

(e) None of the above

17. The equation $3y^2 + y + 4 = 2(6x^2 + y + 2)$ where $y = 2x$ is satisfied by:

(a) no value of x

(b) all values of x

(c) $\star x = 0$ only

(d) all integral values of x only

(e) None of the above

18. If $x < a < 0$ are real numbers then:

(a) $x^2 < ax < 0$

(b) $\star x^2 > ax > a^2$

(c) $x^2 < a^2 < 0$

(d) $x^2 > a^2$ but $a^2 < 0$

(e) None of the above

19. A square, with an area of 40, is inscribed in a semi-circle. The area of a square that could be inscribed in the entire circle with the same radius is:

(a) 80 (b) $\star 100$

(c) 120 (d) 160

(e) None of the above

20. Five times Hannah's money added to Darby's money is more than \$51.00. Three times Hannah's money minus Darby's money is \$21.00. If h represents Hannah's money (in dollars) and d represents Darby's money (in dollars), then:

(a) $\star h > 9$, $d > 6$

(b) $h > 9$, $d < 6$

(c) $h > 9$, $d = 6$

(d) $h > 9$, but we can put no bounds on d.

(e) None of the above

21. Let $T_{h,k}(x, y) = (x+h, y+k)$. Find $T_{h,k}(1, 6)$ if $T_{h,k}(4, 1) = (0, -7)$.

(a) $(-3, 2)$ (b) $(3, -2)$

(c) $\star(-3, -2)$ (d) $(3, 2)$

(e) None of the above

22. There is a linear relationship between the Fahrenheit and Celsius temperature scales. What temperature Celsius is the same as 55 Fahrenheit, to the nearest degree?

(a) $\star 13$ (b) 15

(c) 17 (d) 19

(e) None of the above

23. How many different signals, each consisting of six flags hung in a vertical line, can be formed from four identical red flags and two identical blue flags?

(a) 12 (b) $\star 15$

(c) 18 (d) 21

(e) None of the above

24. If f is a polynomial function of degree 3 with roots at 1, 2, and 3, then a valid statement concerning f is:

(a) $f(0) \cdot f(4) = 0$ (b) $f(0) \cdot f(4) > 0$

(c) $\star f(0) \cdot f(4) < 0$ (d) $f(0) + f(4)$ doesn't exist

(e) None of the above

25. Let $P = (a, b)$ be any point in the plane. The coordinates of the reflection image of P with respect to the line $y = -1$ are:

(a) $\star(a, -b-2)$ (b) $(-2-a, b)$

(c) $(a+1, -b)$ (d) $(a, b+2)$

(e) None of the above

26. The length of the line segment joining the points $(-4, 1, 5)$ and $(4, -7, 3)$ is:

(a) $2\sqrt{23}$ (b) $\star 2\sqrt{33}$

(c) $2\sqrt{14}$ (d) $2\sqrt{10}$

(e) None of the above

27. Bill belongs to a committee which consists of 4 females and 3 males, including him. A subcommittee of 3 members is to be selected, and the subcommittee must contain both sexes. In how many ways can the subcommittee be selected, if Bill is to be on the subcommittee?

(a) 5 (b) 7

(c) 9 (d) 11

(e) None of the above

28. There are 12 points including A in a plane. No three of the points are collinear. How many triangles are there determined by these points (as vertices) which have A as a vertex?

(a) 40 (b) 50

(c) $\star 55$ (d) 220

(e) None of the above

29. The largest real root of $f(t) = t^4 - 3t^3 + 6t^2 + 25t - 39$ has the form $\frac{-1+a}{2}$ where $a > 0$. Given that $2 - 3i$ is a root of this polynomial, find a.

(a) $\sqrt{10}$ (b) $\sqrt{11}$

(c) $\sqrt{12}$ (d) $\star\sqrt{13}$

(e) None of the above

30. The equation of the perpendicular bisector of the base of the isosceles triangle whose vertices are $(1, 1)$, $(10, 2)$, and $(6, -3)$ is written in the form $ax + by + c = 0$. Find $\frac{b}{a}$.

(a) $\frac{1}{6}$ (b) $\frac{1}{76}$

(c) $\frac{1}{8}$ (d) $\star\frac{1}{9}$

(e) None of the above

31. The number of roots of $f(x) = \frac{6(x^2+3)^2 - 6x(4x)(x^2+3)}{(x^2+3)^4}$ is:

(a) None (b) 1

(c) $\star 2$ (d) 3

(e) None of the above

32. The base of a hemisphere of radius 2 inches fits exactly on the base of a cone of height 2 inches. The volume of the solid thus formed, in cubic inches, is:

(a) $\frac{20\pi}{3}$ (b) $\star 8\pi$

(c) $\frac{40\pi}{3}$ (d) 16π

(e) None of the above

33. Given that $f((x-1)^{-1}) = x^{-1}$, then $f(x)$ is:

(a) $(x-1)^{-1}$ (b) $\star\frac{x}{x+1}$

(c) $\frac{x+1}{x}$ (d) $\frac{1}{x} - x$

(e) None of the above

34. If three of the roots of $x^4 + ax^2 + bx + c = 0$ are 1, 2, and 3, then the value of $a + c$ is:

(a) 35 (b) 24

(c) -12 (d) $\star -61$

(e) None of the above

35. The sum of the distances from one vertex of a square of side length two to the midpoints of each of the sides of the square is:

(a) $2\sqrt{5}$ (b) $2 + \sqrt{3}$

(c) $2 + 2\sqrt{3}$ (d) $2 + \sqrt{5}$

(e) None of the above

36. In triangle ABC, D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{DB}$; and F is the midpoint of $\overline{BC}$. If the area of $\triangle ABC$ is 96, then the area of $\triangle AEF$ is:

(a) 16 (b) 24

(c) 32 (d) $\star 36$

(e) None of the above

37. How many pairs of non-zero real numbers (a, b) satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$?

(a) $\star 0$ (b) 1

(c) 2 (d) one pair for each $b \neq 0$

(e) None of the above

38. For which non-zero real numbers x is $\frac{|x-|x||}{x}$ a positive integer?

(a) for negative x only

(b) for positive x only

(c) for all non-zero real numbers x

(d) $\star$for no non-zero real numbers x

(e) None of the above

39. How many integers with four different digits are there between $1,000$ and $9,999$ such that the absolute value of the difference between the first digit and the last digit is 2?

(a) 672 (b) 784

(c) $\star 840$ (d) 896

(e) None of the above

40. The number of pairs of real numbers (x, y) satisfying $x^2 + y^2 = x^3$ is:

(a) 0 (b) 1

(c) 2 (d) $\star$not finite

(e) None of the above

Bonus Questions

41. How many regular polygons have the same number of diagonals as sides? Find, with proof, all of them.

42. Given $\triangle ABC$ with $AB < AC$, if D is a point on $\overline{BC}$ between B and C, prove that $AD < AC$.