Furman University

Calculus Readiness Examination
Sample Questions and Solutions

Department of
Mathematics

Directory

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1. Introduction

The following problems are organized according to broad categories. Try each of the problems. You can see the answer and solution to the problem by clicking on the word "problem" in the green text. When viewing a solution, click on the green arrow to return to the corresponding problem.

2. Solving equations

Problem 2.1. Solve for u in terms of x and y in the equation

$$6(x+2y)^2u + 3y^3 + 9xy^2u = 0$$

Problem 2.2. Find the solutions of the equation

$$3x^2 - 2x = 5$$

3. Exponents

The laws of exponents can be summarized as follows: for real numbers a, b, p, and q,

$$a^p a^q = a^{p+q} \tag{1}$$

$$(a^p)^q = a^{pq} \tag{2}$$

$$a^p b^p = (ab)^p \tag{3}$$

$$a^{-p} = 1/a^p, \tag{4}$$

provided, of course, that the expressions are defined. For example, we can use rule (1) to conclude that $2^6\,2^{-2} = 2^4 = 16$; we can use rule (2) to conclude that $(4^6)^{1/2} = 4^3 = 64$; we can use rule (3) to conclude that

$$\sqrt{3}\sqrt{12} = (3)^{1/2}(12)^{1/2} = (36)^{1/2} = 6;$$

and, finally, we can use rule (4) to conclude that $4^{-1/2} = 1/4^{1/2} = 1/2$. Often you will be required to use these rules in combination.

Problem 3.1. Express

$$2^2 \cdot 3^{-3} + 4^{3/2} \cdot 5^{-1}$$

as a simple fraction.

Problem 3.2. If $\left(x^4\right)^q = x^3$ for all x, then find q.

Problem 3.3. Simplify

$$\left(\frac{a^3 b^{-3} c^{-1}}{a b^4 c^{-3}}\right)^{-1}$$

by expressing it as a product of positive powers of a, b, and c.

Problem 3.4. Simplify and reduce the product of radicals

$$\sqrt{2a^5b^8}\sqrt{8a^3b^3}$$

4. Problems involving common denominators

Problem 4.1. Express the compound fraction

$$\frac{\frac{2}{3}+\frac{1}{12}}{\frac{5}{16}-\frac{1}{24}}$$

as a simple fraction.

Problem 4.2. Use the least common multiple to preform the indicated addition. Simplify your answer.

$$\frac{2}{x^2-1}-\frac{2}{x-1}.$$

Problem 4.3. Assuming $x \neq 2$, simplify the expression

$$\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$$

5. Completing the square

The key to understanding completing the square is to note that

$$(x \pm a/2)^2 = x^2 \pm ax + (a/2)^2.$$

Thus, if we are confronted with the quadratic expression of the form $x^2 \pm ax$, we can transform it into a perfect square by adding $(a/2)^2$. To retain equality, we must compensate for the addition of this term by either (a) simultaneously subtracting it, or (b) simultaneously adding it to the other side of the equation. For example, to complete the square in the equation

$$y = x^2 + 6x + 5,$$

I need to add 9 to both sides of the equation. The 9 that is added to the right-hand side will be grouped with $x^2 + 6x$ to form the perfect square.

$$y + 9 = (x^2 + 6x + 9) + 5$$

Now we recognize the perfect square:

$$y + 9 = (x + 3)^2 + 5.$$

Depending upon what you are going to do, you can simplify this equation further still.

Problem 5.1. Find the vertex of the parabola with equation $y = x^2 - 8x + 13$.

6. Lines

The most useful formula for the equation of a line in the plane is the so-called *point-slope formula*. If a line has slope m and passes through the point (h, k), then the equation of the line is given by

$$(y - k) = m(x - k)$$

The equation

$$y = mx + b$$

is the celebrated *slope-intercept form* of the equation of the line. The slope-intercept form allows us to immediately determine the slope m and the y-intercept $(0, b)$ of the line.

There are two additional geometric facts about lines that are useful: the slopes of parallel lines are equal; the slopes of perpendicular lines multiply to -1.

Problem 6.1. Find the equation of the line passing through $(3/2, 1)$ and $(4, 2/3)$.

Problem 6.2. Find the equation of the line parallel to $3x + 4y = 24$, passing through $(2, 1)$.

Problem 6.3. Find the equation of the line perpendicular to $2x + 7y = 13$ passing through the y intercept of $4x + 5y = 80$.

7. Trigonometry

Problem 7.1. If $\sin(x) = 12/13$ and $90° < x < 180°$, then what is $\cos(x)$?

Problem 7.2. If θ is an acute angle in a right triangle and $\cos(\theta) = 1/10$, find $\tan(\theta)$.

8. Inequalities

Problem 8.1. Solve the inequality $-2 \le 8 - 5x \le 10$.

Problem 8.2. Solve the inequality

$$\frac{x-1}{2-x} > 3.$$

9. Absolute values

Problem 9.1. Evaluate $||5-7|-10|$.

Problem 9.2. Find all x such that $||x|-8| = 5$.

Problem 9.3. Simplify $(x^2-1)/|x-1|$ subject to the condition that $x < 1$.

Solutions to Exercises

Problem 2.1. This is a linear equation in u. We begin by isolating the terms involving u on (say) the left-hand side of the equation:

$$6(x + 2y)^2u + 9xy^2u = -3y^3.$$

Factoring u on the left-hand side produces the equation

$$\left(6(x + 2y)^2 + 9xy^2\right)u = -3y^3$$

Finally, we solve for u by division:

$$u = \frac{-3y^3}{6(x + 2y)^2 + 9xy^2}$$

◀

Problem 2.2. This is a quadratic equation in x. First we write the equation in standard quadratic form:

$$3x^2 - 2x - 5 = 0$$

According to the quadratic formula, the solution(s) to this equation are of the form

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(-5)}}{2 \cdot 3}.$$

After some numerical work, we see that the two solutions are $x = -1$ and $x = 5/3$.

This problem could have been solved by factoring. Observe that

$$3x^2 - 2x - 5 = (3x - 5)(x + 1) = 0$$

has solutions corresponding to $3x - 5 = 0$ and $x + 1 = 0$, or $x = 5/3$ and $x = -1$ respectively. ◀

Problem 3.1. By rule (4),

$$2^2 \cdot 3^{-3} = 2^2/3^3 = 4/27.$$

By rule (2),

$$4^{3/2} = \left(4^{1/2}\right)^3 = 2^3 = 8.$$

Thus, by another application of rule (4), $4^{3/2} \cdot 5^{-1} = 8/5$. Finally the required number is

$$\frac{4}{27} + \frac{8}{5} = \frac{236}{135}.$$

◀

Problem 3.2. By rule (2), $\left(x^4\right)^q = x^{4q}$. Since $x^{4q} = x^3$ for all x, the exponents must be equal, and we may conclude that $4q = 3$ or $q = 3/4$. ◀

Problem 3.3. We will work inside of the parenthesis first. Using the rules (1) and (4), we see that

$$\frac{a^3b^{-3}c^{-1}}{ab^4c^{-3}} = \frac{a^2c^2}{b^7}.$$

It follows that

$$\left(\frac{a^3b^{-3}c^{-1}}{ab^4c^{-3}}\right)^{-1} = \left(\frac{a^2c^2}{b^7}\right)^{-1} = \frac{b^7}{a^2c^2}.$$

◀

Problem 3.4. Using rules (3) and (1), we see that

$$\begin{aligned}\sqrt{2a^5b^8}\sqrt{8a^3b^3} &= \sqrt{(2a^5b^8)(8a^3b^3)}\\ &= \sqrt{16a^8b^{11}}\\ &= 4a^4b^5\sqrt{b}\end{aligned}$$

◀

Problem 4.1. There are several ways to accomplish this. The least common multiple of the denominators 3, 12, 16, and 24 is 48. Multiplying the numerator and denominator of the compound fraction by 48 reveals

$$\begin{aligned}
\frac{\frac{2}{3}+\frac{1}{12}}{\frac{5}{16}-\frac{1}{24}} &= \frac{\left(\frac{2}{3}+\frac{1}{12}\right)48}{\left(\frac{5}{16}-\frac{1}{24}\right)48} \\
&= \frac{\frac{2}{3}48+\frac{1}{12}48}{\frac{5}{16}48-\frac{1}{24}48} \\
&= \frac{32+4}{15-2} \\
&= \frac{36}{13}
\end{aligned}$$

◀

Problem 4.2. A common denominator can be obtained by finding the least common multiple of $x^2 - 1$ and $x - 1$. Notice that $x^2 - 1 = (x - 1)(x + 1)$; thus, the least common multiple of $x^2 - 1$ and $x - 1$ is $x^2 - 1$. The second term must be altered accordingly to obtain the common denominator. Here is the required algebra:

$$\begin{aligned} \frac{2}{x^2 - 1} - \frac{2}{x - 1} &= \frac{2}{x^2 - 1} - \left(\frac{2}{x - 1}\right)\left(\frac{x + 1}{x + 1}\right) \\ &= \frac{2}{x^2 - 1} - \frac{2(x + 1)}{x^2 - 1} \\ &= \frac{2 - 2(x + 1)}{x^2 - 1} \\ &= \frac{-2x}{x^2 - 1} \end{aligned}$$

◀

Problem 4.3. If we write $x - 2$ as $(x-2)/1$, then the least common multiple of the denominators x, 2, and 1 is $2x$. We can simplify the compound fraction by multiplying the numerator and denominator by $2x$. Here is the relevant algebra:

$$\begin{aligned}\frac{\dfrac{1}{x} - \dfrac{1}{2}}{x-2} &= \frac{\left(\dfrac{1}{x} - \dfrac{1}{2}\right) 2x}{(x-2)2x} \\ &= \frac{2-x}{(x-2)(2x)} \\ &= -\frac{1}{2x},\end{aligned}$$

where we have used the fact that $2 - x = -(x-2)$ to effect the cancellation. ◀

Problem 5.1. To find the vertex, we need to complete the square in the variable x. Thus we have

$$y + 16 = (x^2 - 8x + 16) + 13$$

Recognizing the perfect square, this equation can be rendered as

$$y = (x - 4)^2 - 3$$

It is easy to see that this is the equation of a parabola which opens upwards. The minimum value of the parabola will occur when $x = 4$, at which point $y = -3$. The coordinates of the vertex are $(4, -3)$.

◀

Problem 6.1. First we compute the slope of the line:

$$m = \frac{1 - 2/3}{3/2 - 4} = -\frac{2}{15}.$$

In point-slope form, the line may be expressed as

$$y - 1 = -\frac{2}{15}(x - 3/2),$$

which can be expressed in slope-intercept form as

$$y = -\frac{2}{15}x + \frac{6}{5}.$$

◀

Problem 6.2. Since parallel lines have the same slope, our first task is to determine the slope of the given line. Solving for y, we obtain

$$y = -34x + 6$$

The slope of the given line is $-3/4$. The parallel line must be given by the equation

$$y - 1 = -\frac{3}{4}(x - 2).$$

◀

Problem 6.3. Since the first line may be expressed as

$$y = -\frac{2}{7}x + \frac{13}{7}$$

our perpendicular line must have slope 7/2. The y-intercept of $4x + 5y = 80$ can be found by setting $x = 0$ and solving for y. This yields $y = 16$. The y-intercept is thus $(0, 16)$. The equation of the perpendicular line is thus

$$y = \frac{7}{2}x + 16.$$

◀

Problem 7.1. Since $\sin^2(x) + \cos^2(x) = 1$, it must be that

$$\cos(x) = \pm\sqrt{1 - \left(\frac{12}{13}\right)^2} = \pm\frac{5}{13}.$$

Of course, this doesn't solve the problem, since the sign of the $\cos(x)$ has not been resolved. However, since $90° < x < 180°$, $\cos(x)$ is a negative number; thus,

$$\cos(x) = -\frac{5}{13}$$

◀

Problem 7.2. As in the previous problem, we can determine $\sin(\theta)$ through the Pythagorean identity.

$$\sin(\theta) = \pm\sqrt{1 - (1/10)^2} = \pm\frac{3\sqrt{11}}{10}.$$

Since θ is an acute angle, $\sin(\theta)$ is positive, and

$$\sin(\theta) = \frac{3\sqrt{11}}{10}.$$

It follows that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = 3\sqrt{11}.$$

◀

Problem 8.1. We can solve inequalities much like equalities. There is one special rule: the sign of the inequality is reversed upon dividing or multiplying by a negative number.

For the problem at hand, we have

$$\begin{aligned} -2 \leq 8 - 5x &\leq 10 \\ -10 \leq -5x &\leq 2 \\ 2 \geq x &\geq -2/5 \end{aligned}$$

The answer should be expressed as a set. The solution set is $[-2/5, 2]$ in interval notation or $\{x : -2/5 \leq x \leq 2\}$ in set-builder notation.

◀

Problem 8.2. It helps to transform the inequality. The given inequality is equivalent to

$$\frac{x-1}{2-x} - 3 > 0 \qquad \text{or} \qquad \frac{4x-7}{2-x} > 0.$$

The numerator is equal to at $7/4$ and the denominator is equal to 0 at 2. Thus the line is divided into three intervals: $(-\infty, 7/4)$, $(7/4, 2)$, and $(2, \infty)$. On the first interval, the numerator is negative and the denominator is positive; thus the ratio is negative. We reject this interval. On the second interval, both numerator and denominator are positive; thus the ratio is positive. We accept this interval. On the third interval, the numerator is postive and the denominator is negative; thus, the ratio is negative. We reject this interval. The solution is thus $(7/4, 2)$. ◀

Problem 9.1. The answer is 8. ◀

Problem 9.2. If $||x| - 8| = 5$, then $|x| - 8 = \pm 5$, that is, $|x| = 13$ or $|x| = 3$. Thus $x = +13, -13, 3, -3$. ◀

Problem 9.3. If $x < 1$, then $|x-1| = -(x-1)$ and

$$\frac{x^2-1}{|x-1|} = \frac{x^2-1}{-(x-1)} = -\frac{(x-1)(x+1)}{(x-1)} = -(x+1).$$

◀