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ON THE DEAD END DEPTH OF THOMPSON'S GROUP F

JUSTIN HALVERSON

Abstract. Thompson's group F was introduced by Richard Thompson in the 1960's and has since found applications in many areas of mathematics including algebra, logic and topology. We focus on the dead end depth of F, which is the minimal integer N such that for any group element, g, there is guaranteed to exist a path of length at most N in the Cayley graph of F leading from g to a point farther from the identity than g is. By viewing F as a diagram group, we improve the greatest known lower bound for the dead end depth of F with respect to the standard consecutive generating sets.

1. Introduction

The dead end depth of a group with respect to a finite generating set S is the minimal integer N such that such that for any group element, g, the distance from g to the complement of the ball of radius $d(e, g)$ centered at the identity, e, is at most N. Among other places, dead ends have found application in the proof in [10] demonstrating a random walk that is biased towards the identity on the lamplighter group but that escapes from the identity faster than a simple random walk. Dead ends also played a role in Bogopol′skiĭ's result that infinite commensurable hyperbolic groups are bi-Lipschitz equivalent [1].

A common theme in Geometric Group Theory is to classify the generating sets with respect to which a certain group or class of groups possess a given property. For dead end depth, few definitive results of this kind are known. Bogopol′skiĭ [1] proved that the depth of a hyperbolic group with respect to any finite generating set is finite. Lenhart [9] has shown abelian groups and groups with more than one end in the sense of Hopf and Freudenthal have bounded dead end depth. In the other direction, Cleary and Riley [3, 4] construct a finitely presented group with unbounded dead end depth with respect to a particular generating set. And, Riley and Warshall [11] have shown that having unbounded dead end depth is not a group invariant, that is, that is it depends on the choice of generating set.

For Thompson's group F, the exact dead end depth is known only for the standard generating set, $\{x_0, x_1\}$, in which case the depth is known to be 3 [5]. For the larger standard consecutive generating sets, $X_n = \{x_0, x_1, \cdots, x_n\}$, the depth is known only to be bounded between $\frac{n}{2}$ and $4n - 2$ [8]. In this paper we prove the following theorem raising the known lower bound for the dead end depth of F with respect to X_n to $2n - 7$.

Main Theorem. *For $n \geq 4$, The dead end depth of Thompson's group F with respect to the generating set $X_n = \{x_0, x_1, \ldots, x_n\}$ is at least $2n - 7$.*

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This paper is organized as follows: In the next section we formally define Thompson's Group F and review some necessary background information and results about F. In that section, we also formally define the concept of dead end depth. Section 3 is devoted to developing the methods and tools necessary to prove our main theorem and to proving the theorem assuming the lemmas of that section. The fourth section holds the proofs to the lemmas introduced in the third section, which are somewhat technical.

2. Background

2.1. **Introduction to Thompson's Group F.** We now give a brief introduction to Thompson's Group F. For a more detailed explanation, we refer the reader to [2]. As a set, Thompson's Group F is the set of orientation preserving piecewise linear homeomorphisms, f, from the unit interval I to itself such that:

1. There are only finitely many points of non-differentiability of f,
2. Each point of non-differentiability of f occurs at a dyadic rational number,
3. Every slope of f is a power of 2.

The group operation of F is composition of functions.

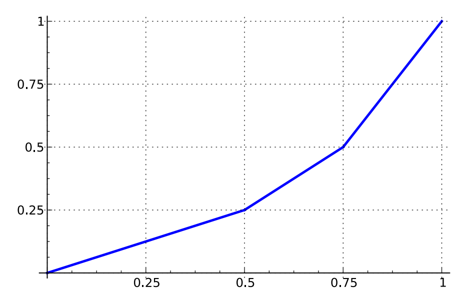
The group F is usually studied by the following infinite and finite presentations.

$$F = \langle x_k, k \geq 0 \mid x_i^{-1} x_j x_i = x_{j+1} \text{ if } i < j \rangle \tag{1}$$

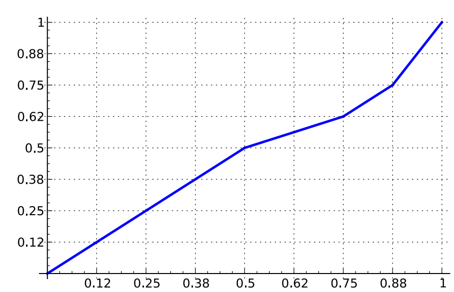
$$F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle \tag{2}$$

The elements x_0 and x_1 are the functions given as follows with their graphs.

$$x_0(t) = \begin{cases} \frac{1}{2}t, & 0 \leq t \leq \frac{1}{2} \\ t - \frac{1}{4}, & \frac{1}{2} \leq t \leq \frac{3}{4} \\ 2t - 1, & \frac{3}{4} \leq t \leq 1 \end{cases}$$



$$x_1(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2}t + \frac{1}{4}, & \frac{1}{2} \leq t \leq \frac{3}{4} \\ t - \frac{1}{8}, & \frac{3}{4} \leq t \leq \frac{7}{8} \\ 2t - 1, & \frac{7}{8} \leq t \leq 1 \end{cases}$$



Note that the graph of x_1 consists of the graph of the identity on the first half of I together with a copy of of the graph of x_0 that has been "scaled down" by a factor one half and placed in the upper right hand corner. This means that one can think of x_1 as "acting as x_0" on the subinterval $[\frac{1}{2}, 1]$ of I and as the identity elsewhere.

For $n \geq 1$, the generating sets $X_n := \{x_0, x_1, x_2, \ldots, x_n\}$ are referred to as the *standard consecutive generating sets* and $X_\infty = \{x_0, x_1, x_1, \ldots\}$ is referred to as the *standard infinite generating set* for F. With respect to X_∞, each element g of F can be expressed uniquely in *normal form*,

$$g = x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k} x_{j_l}^{-s_l} \ldots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

with $s_i, r_i > 0$, $i_1 < i_2 < \cdots < i_k$, and $j_1 < j_2 < \cdots < j_l$ and such that if both x_i and x_i^{-1} appear in the expression then so does x_j or x_j^{-1} for some $j > i$ (see [2]). The normal form is an important tool in the study of the combinatorial properties of F. One important property of the normal form is that it is a minimal length expression for g with respect to X_∞. The normal form also provides a canonical way to decompose an element g as the product of a "positive" element and a "negative" element. Explicitly, if the normal form representation of g is, $g = x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k} x_{j_l}^{-s_l} \ldots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$, then we write $g_+ = x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k} x_{j_l}^{-s_l}$ and $g_- = x_{j_2}^{-s_2} x_{j_1}^{-s_1}$, and call g_+ the "positive" portion of g and g_- the "negative" portion of g.

2.2. **Diagram representations of elements of F.** Because it is difficult to understand the effect of composing two elements of F by studying their piecewise formulas, elements of F are frequently represented combinatorially. One combinatorial representation for elements of F is the *infinite diagrams* described in [6] and [7], whose definition we now review.

Definition 2.1. *An* infinite diagram *is a directed planar graph D with infinite vertex set $\{v_0, v_1, v_2, \ldots\}$ together with an embedding of D in the plane in such a way that:*

- *For each i, there is an edge directed from v_i to v_{i+1}.*
- *The v_i are discrete points of the x-axis and the edge from v_i to v_{i+1} is smoothly embedded in the x-axis. Such edges are called* central edges*.*
- *All other edges are smoothly embedded in the plane either entirely above or entirely below the x-axis and are directed left to right.*
- *All finite regions have three edges. Such regions are called* cells*.*
- *There are only a finite number of non-central edges in D.*

Since a diagram has only finitely many non-central edges, there is a rightmost vertex v_k incident to a non-central edge. We call the subgraph of D spanned by vertices $\{v_1, v_2, \ldots, v_k\}$ the *essential* portion of D. Generally, we omit the infinite right tail of a diagram from figures and display only the essential portion of the diagram in questions. Figure 1 shows an example of the essential portion of a diagram, with directions on the edges omitted. We require some specialized graph-theoretical terminology specific to diagrams, which we now describe. The initial and terminal vertices of edge e are denoted by $i(e)$ and $t(e)$ respectively. An *upper* (respectively *lower*) edge is an edge which lies completely above (respectively below) the central line of a diagram. The lower edge e_1 is *below* the lower or central edge e_2 if $i(e_1)$ is equal to or left of $i(e_2)$ and $t(e_1)$ is equal to or to the right of $t(e_2)$ on the x-axis. Similarly, the upper edge e_1 is *above* the upper or central edge e_2 if the same condition on the endpoints of e_1 and e_2 holds. A lower or central edge e is *exposed from the bottom* if there is no edge below e and the upper or central edge e is *exposed from the top* if there is no edge above e. The *bottom path* of D is the path consisting of the set of central or lower edges that are exposed from the bottom and the *upper path* is the path consisting of the set of upper or central edges exposed from the top. Vertices on the bottom (respectively top) path are *exposed from the bottom* (respectively *top*). A cell C lying above the x-axis is an *exposed upper cell* if two of its edges are central edges that are exposed from the bottom. Vertices v_1, v_2 and v_3 are the vertices of an exposed cell in Figure 1.

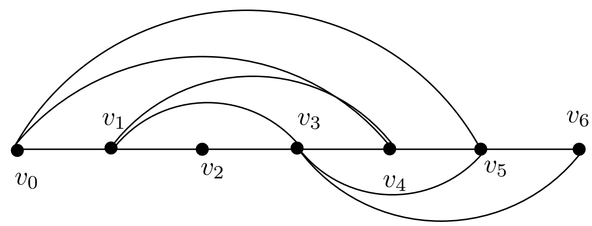


FIGURE 1. Essential portion of a diagram with upper exposed cell

Note that the two central edges of an exposed upper cell are necessarily embedded adjacent to each other on the x-axis. Finally a vertex v is said to be *covered* by the bottom edge e if $i(e)$ is to the left of v and $t(e)$ is to the right of v.

In order to establish a one-to-one correspondence between infinite diagrams and elements of F, one must restrict attention to the reduced diagrams, which we now define.

Definition 2.2. *A* dipole *in the infinite diagram D is a subgraph D_0 of D consisting of three vertices, v_i, v_{i+1} and v_{i+2} together with the central edges between them, an upper edge from v_i to v_{i+2} and also a lower edge from v_i to v_{i+2} as shown in Figure 2.*

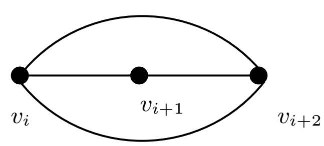


FIGURE 2. A dipole

Definition 2.3. *The diagram D is* reduced *if it contains no subgraph that is a dipole.*

Two infinite diagrams D_1 and D_2 are *isomorphic* if there is an orientation-preserving homeomorphism of $\mathbb{R}^2$ to itself that restricts to a direction-preserving graph isomorphism between D_1 and D_2 taking central edges to central edges, upper edges to upper edges and lower edges to lower edges. Since it is the set of isomorphism classes of reduced diagrams that is in one-to-one correspondence with elements of F, it is convenient to abuse notation and refer to the "diagram D" when one really means the "isomorphism class of diagram D". For the remainder of the paper, we adopt the convention of simply referring to "diagrams" instead of "isomorphism classes of diagrams". Additionally, for the remainder of the paper, all diagrams under consideration will be infinite, reduced diagrams, but we omit the repetition of the words "infinite" and "reduced". Henceforth, all diagrams are assumed to be infinite reduced diagrams.

The correspondence between the set of reduced diagrams and F is most easily defined by giving each vertex of a diagram D an "upper" and "lower" label, defined as follows. Let $w_0, w_1, w_2, \ldots$ be the vertices of the bottom path in order from left to right and let $u_0, u_1, u_2, \ldots$ be the vertices of the top path in order from left to right. Vertex w_i has bottom label $\frac{2^i-1}{2^i}$ and vertex u_i has top label $\frac{2^i-1}{2^i}$. The bottom labels of the remaining vertices without bottom labels are defined by the property that if C is a lower cell with vertices v_i, v_j and v_k with $i < j < k$ then the bottom label of v_j is the average of the bottom labels of v_i and v_k. The top labels of the remaining vertices without top labels are defined similarly. Figure 3 illustrates the labeling of the essential portion of a diagram.

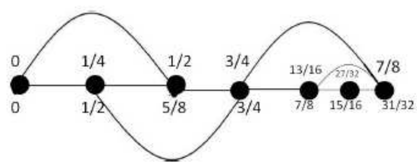


FIGURE 3. A labeled diagram

We are now in a position to define the correspondence between the set of (isomorphism classes of reduced, infinite) diagrams and F. For diagram D, denote by $top(v)$ and $bot(v)$ the top and bottom labels of vertex v. The orientation preserving piecewise linear homeomorphism of I with itself that corresponds to D is the function f_D defined by:

1. $f_D(bot(v)) = top(v)$ for every vertex v of D
2. f_D is linear on the complement of the set of bottom labels of D
3. f_D is an orientation preserving piecewise linear homeomorphism from I to itself.

For example, the function given by the diagram in Figure 3 maps the interval $[0, \frac{1}{2}]$ linearly to $[0, \frac{1}{4}]$, the interval $[\frac{1}{2}, \frac{5}{8}]$ linearly to the interval $[\frac{1}{4}, \frac{1}{2}]$, and so on. The (isomorphism class of reduced) diagram corresponding to an element of F in this way is unique, and we denote the diagram corresponding to g by $D(g)$.

The diagram representations of the elements of x_0 and x_1 are shown in Figure 4. As usual, we omit the infinite right tail of central vertices and show only the essential portion of the diagrams.

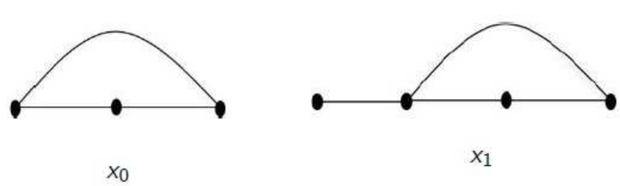


FIGURE 4. Diagrams for x_0 and x_1

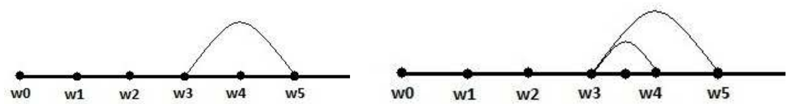


FIGURE 5. Multiplication by x_3 resulting in addition of an edge

In general, x_i corresponds to the diagram with a single upper edge from vertex v_i to v_{i+2}, and the diagram for x_i^{-1} is constructed by reflecting the diagram for x_i about the x-axis.

An important relationship between the diagram for an element and its normal form is the fact that if we consider α_+ and α_- as elements of F in their own rights, then the diagram for α_+ is the diagram consisting of the upper and central edges from the diagram of α and the diagram of α_- is the diagram consisting of central and lower edges of the diagram of α. Thus, the number of upper (respectively lower) edges in the diagram for g_n is equal to the number of positive (respectively negative) generators (counted with multiplicity given by their exponents) occurring in the normal form of g.

For the purposes of this paper, we need only to multiply general elements of F by single generators in X_n at a time. In order to see the effect of multiplication by a generator, consider an element g in F with diagram $D(g)$. Label by $w_0, w_1, w_2, \ldots$ the vertices along the bottom path of g starting with the vertex labeled 0 on top and bottom. Now consider a generator $x_i \in X_n$.

To produce the diagram for $g \cdot x_i$, proceed as follows:

- If there exists one or more lower edges directed out of w_i, remove the bottom-most edge.
- If there is no lower edge directed out of w_i, alter $D(g)$ by adding an upper edge out of w_i, terminating at w_{i+1} below all upper edges and subdividing the central edge out of w_i into two central edges, one originating at w_i terminating at a new vertex w and one originating at the new vertex w and terminating at w_{i+1}.

Figure 5 illustrates the result of the modification in the second case with x_3.

To get the diagram for $g \cdot x_i^{-1}$, modify the diagram for g as follows:

- If w_i is the first vertex of an exposed upper cell, eliminate the three edges of that cell and the central vertex it contains, and add a new central edge connecting w_i and w_{i+2}.
- If w_i is not the initial vertex of an exposed cell, add a bottom-most lower edge initiating at w_i and terminating at w_{i+2}.

2.3. **Dead Ends.** The length of an element $g \neq id$ of a group G with respect to a generating set S is the minimal length of a product of elements from $S \bigcup S^{-1}$ that is equal to g. Geometrically, the length of g is the distance from the identity element to g in the Cayley Graph of G with respect to the generating set S. We denote by $l_S(g)$ the length of g with respect to the generating set S. If a there is a fixed generating set S under consideration we occasionally omit the mention of the generating set and write simply $l(g)$ for $l_S(g)$. We now define dead ends.

Definition 2.4. *Let G be an infinite group, and let S be a finite generating set for G. The element $g \in G$ is a dead end element with respect to S if:*

$$l(gs) \leq l(g) \forall\ s \in (S \bigcup S^{-1}).$$

Definition 2.5. *Let G be a group, and let S be a finite generating set for G. The* dead end depth *of element g in G is given by,*

$$depth(g) := \min\{l(h) \mid l(gh) > l(g)\}.$$

We remark that with this definition, an element that is not a dead end has dead end depth 1, and a dead end has depth at least 2. This is consistent with the convention in, for example, [3] and [5] but is inconsistent with the convention in [8], where the dead end depth of an element is one less than the depth given here.

Definition 2.6. *The* dead end depth *of finitely generated group G with respect to the finite generating set S is the maximum N that is the dead end depth of an element in G if such an N exists, and is infinity otherwise.*

2.4. **Calculating Length in F.** Dead ends are known to exist in Thompson's Group F with respect to any consecutive generating set X_n [5, 8]. As mentioned in the introduction, the exact dead end depth of F is known only with respect to X_1, and for $n \geq 3$ the dead end depth is known only to be bounded between $\frac{n}{2}$ and $4n-2$. Using infinite diagrams to analyze the effect on length of multiplication by a generator, we improve the lower bound by proving the following Theorem.

Theorem 2.1. *For $n \geq 4$, The dead end depth of Thompson's group F with respect to the generating set $X_n = \{x_0, x_1, \ldots, x_n\}$ is at least $2n-7$.*

Our main tool is the formula in [8] for determining the length of an element of F with respect to X_n, which reads as follows.

Theorem 2.2 ([8] Theorem 3.3)**.** *For every $g \in F$, the word length of g with respect to the generating set X_n is given by the formula,*

$$l_n(g) = l_\infty(g) + 2P_n(g)$$

where $l_\infty(g)$ is the total number of upper and lower edges of the diagram for g, and $P_n(g)$ is the penalty weight of g.

As mentioned during the discussion of diagrams, the first component of the word length formula, $l_\infty(g)$, is the word length of the element g with respect to the infinite generating set X_∞, which is also the number of generators appearing in the normal form representation of g, and the number of non-central edges in $D(g)$. The second component, $P_n(g)$, is the so-called penalty weight of g, which we now define.

A vertex in $D(g)$ is of *penalty type* if it is the initial vertex of a non-central edge, or it is a separating vertex whose removal separates $D(g)$ into two components with the right component containing a non-central edge. Such a vertex will be called an "essential cut" vertex. Even though in the strict graph-theoretical sense of the term, any vertex of $D(g)$ to the right of the last non-central edge is a cut vertex, these are not "essential" cut vertices in our definition. Recall that $D(g)$ is a directed graph, with edges directed from left to right. A *penalty tree* in the diagram $D(g)$ is a directed subtree of $D(g)$ rooted at the vertex v_0 that contains a directed path from v_0 to each penalty type vertex of $D(g)$. We note that a penalty tree may contain non-penalty type vertices.

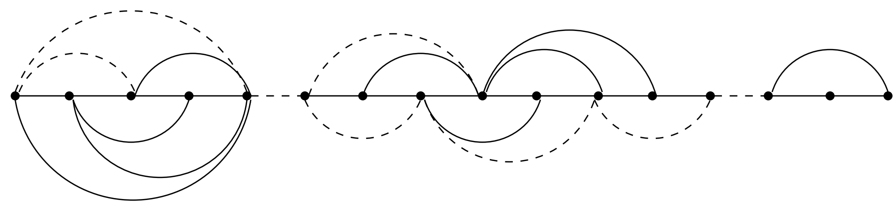


FIGURE 6. Calculating $P_n(g)$

The *penalty weight*, $P_n(T)$ of penalty tree T with respect to X_n is the number of vertices w (penalty type or not) in T satisfying,

1. The distance from v_0 to w through T is greater than or equal to 2,
2. There is a directed path through T of length at least $n-1$ from w to a leaf ℓ of T.

The penalty weight, $P_n(g)$, of g with respect to X_n is the minimal penalty weight of all penalty trees in $D(g)$. That is, $P_n(g) = \min\{P_n(T) \mid T \text{ is a penalty tree of } g\}$

For example consider the element, g, whose diagram is shown in Figure 6. Now, $l_\infty(g) = 15$ and the penalty tree indicated by the dashed edges has penalty weight 3. Thus, our length formula gives us that $l_3(g)$ is at most 21. It is not hard to show that the given T is in fact of minimal weight among all penalty trees for g, so $l_3(g) = 21$.

3. Increased lower bound for dead end depth

In this section we fix an integer $n \geq 4$ and develop the tools used in the proof of our main theorem. The proofs are somewhat technical and we postpone the most technical ones until Section 4.

3.1. **The Element g_n.** The proof that the dead end depth of X with respect X_n is at least $2n-7$ consists simply of exhibiting an element g_n, which we formally define below, whose dead end depth is shown to be at least $2n-7$. The diagram of g_n is constructed by repeating a "principal section" $7n$ times. This section is defined in such a way as to prevent multiplication by any element of F of length less than or equal to $2n-8$ with respect to X_n from altering the top half of the diagram for g_n.

The principal section of g_n is the diagram, denoted by S_n, that has $2^{2n-4}+1$ vertices, $\{s_0, s_1, \ldots, s_{2^{2n-4}}\}$ with central edges from s_i to s_{i+1} for each i. Additionally, there is an upper edge from s_1 to s_3, and for $i \geq 3$ there is an upper edge from s_0 to s_i. The lower edges are constructed recursively as follows:

1. There is a lower edge from s_0 to $s_{2^{2n-4}}$, the "level $2n-4$" lower edge,
2. For every level $k \geq 2$ lower edge edge from vertex s_i to s_j, there is a level $k-1$ lower edge from s_i to $s_{\frac{i+j}{2}}$ and a level $k-1$ lower edge from $s_{\frac{i+j}{2}}$ to s_j,
3. A total of $2n-4$ levels of lower edges are added.

The diagram for the element g_n is constructed by gluing $6n$ copies of the diagram S_n together end to end, as shown in Figure 7. Note that g_n has $2n-4$ levels of lower edges, the first three vertices of each principal section are penalty type vertices, and after that every other vertex of each principal section is a penalty type vertex.

In order to show that the dead end depth of G_n is at least $2n-7$, it is not necessary to establish the exact length of g_n with respect to X_n, but it is necessary

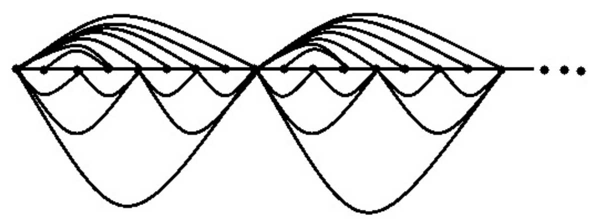


FIGURE 7. Sample g_n Diagram (edges of level less than $2n-2$ not shown)

to find a minimal weight penalty tree. It is not difficult to show that a minimal penalty tree T_n may be constructed without using any lower edge of level greater than 1 as follows. The vertex set of T_n is the set of penalty type vertices of g_n. The first central edge in each principal section of g_n belongs to T_n. The leftmost level 1 lower edge of each principal section belongs to T_n. And, for any penalty type vertex v of g_n that is not the first, second or third vertex of a section, the unique upper edge terminating at v belongs to T_n.

Since it contains all penalty type vertices of g_n, the subtree T_n is a penalty tree for g_n. To get a sense of which vertices of T_n are weighted vertices, note that T consists of one long path consisting of the uppermost edges in the essential portion of the diagram of g_n together with many single edges attached to the essential cut vertices. Therefore, the only weighted vertices in T_n are those essential cut vertices w that are a distance at least $n-2$ from the last essential cut vertex along the upper path of g_n. The distance in the previous sentence is $n-2$ because the edges in T_n originating at this vertex make w at distance at least $n-1$ from a leaf.

Because every penalty tree of g must contain all of the essential cut vertices of g, it is not difficult to show that T_n is actually a minimal penalty tree for g. Moreover, the only non-weighted vertices in T_n that are on the upper path of g_n are v_0, the first vertex of the second principal section and those that are at distance greater than $4n$ from v_0, though not all such verteces are actually weighted. Additionally, multiplication of g_n by an element $\alpha \in F$ of length less than $2n-7$ does not modify any part of the diagram of g_n farther to the right than the $4n^{th}$ principal section. For these two reasons, we call the portion of $D(g_n)$ consisting of the first $4n$ principal sections the *active* portion of $D(g_n)$.

This allows us to prove,

Lemma 1. *Let α be an element of F with $l_n(\alpha) \leq 2n-8$ such that the normal form of α has more positive generators than negative, then $l_n(g_n\alpha) \leq l_n(g_n)$.*

Proof. Let $\alpha = g = x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k} x_{j_l}^{-s_l} \ldots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$ be the normal form for α. Since the normal form of α contains more positive generators than negative generators, $r_1 + r_2 + \cdots + r_k > s_1 + s_2 + \cdots s_l$. Note that α has at most $2n-8$ upper edges total since $l_\infty(\alpha) \leq l_n(\alpha) \leq 2n-8$. We consider the process of constructing the diagram for $g_n\alpha$ from the diagram for g_n by analyzing the effect of multiplying by one generator from the normal form of α at a time starting with x_{i_1}. Since g_n has $2n-4$ levels of lower edges, the m^{th} positive generator from the normal form of α is guaranteed to have the effect of removing one lower edge of level at least 2

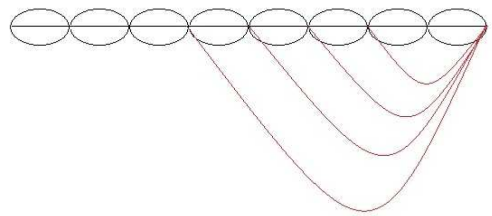


FIGURE 8. Adding edges that do not reduce penalty weight

from the diagram resulting from multiplying by the first $m-1$ positive generators. Thus, $l_\infty(g_n x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k}) = l_\infty(g_n) - (r_1 + r_2 + \cdots r_k)$.

Also, since the removal of at most $2n-8$ lower edges from $D(g)$ cannot expose an upper cell, the m^{th} negative generator from the normal form has the effect of adding one lower edge to the diagram resulting from multiplying by all of the positive generators and the first $m-1$ negative generators. Since $r_1+r_2+\cdots+r_k > s_1 + s_2 + \cdots s_l$, we find that $l_\infty(g_n\alpha) \leq l_\infty(g_n)$.

Now, since α has at most $2n-8$ upper edges, the diagram for $g_n\alpha$ differs from the diagram for g only in lower edges of level greater than 1. Thus, the tree T_n' drawn in the diagram for $g_n\alpha$ exactly as the tree T_n in the diagram for g_n is a valid penalty tree for $g_n\alpha$. Thus, $p_n(g_n\alpha) \leq p_n(g_n)$. Therefore, $l_n(g_n\alpha) = l_\infty(g_n\alpha) + 2p_n(g_n\alpha) \leq l_\infty(g_n) + 2p_n(g_n) = l_n(g_n)$, as required. □

3.2. **Informal Strategy to determine the depth of g_n.** Here we heuristically describe the strategy used to determine the dead end depth of g_n. Informally, we view the situation as follows. No multiplication of g_n by an element α with $l_n(\alpha) \leq 2n-8$ can modify the active section of $D(g_n)$ since such the diagrams of such elements contain at most $2n-8$ edges, all of which originate at vertices closer than $4n$ to v_0. Therefore, multiplication of g_n such an α cannont create a penalty type vertex. For the same reason, none can eliminate an edge of T_n. This means that the only way to increase the length of g_n by multiplying it by an element of length less than $2n-8$ would be to find a way to add a set of edges whose net effect does does not reduce the penalty weight. The general strategy for doing so is displayed in Figure 8. The ovals represent the principal sections of G_n and the additional lines are the added edges.

Recall that all essential cut vertices must belong to every penalty tree T of g_n. Now, if an essential cut vertex of g_n is covered below by an edge added by multiplying g_n by an element of $X_n \cup X_n^{-1}$ then the penalty tree may be modified to eliminate it as a weighted vertex by using the new edge instead of the upper edge terminating at the new edge's endpoint. This modification has the effect of removing the covered vertex from the single long path in T_n and preventing it from being at distance $n-1$ from a leaf. This holds true not only for multiplication of g_n by generators, but for the multiplication of $g_n\alpha$ by generators provided that α added no more than $n-2$ edges to $D(g)$. After $n-2$ edges are added, length can start to increase again. Since the first $n-2$ edges reduced the penalty weight and thus the actual length, a total of $2n-3$ edges would be required to increase length beyond that of g_n. Thus, in order for $l_n(g_n\alpha)$ to be greater than $l_n(g_n)$, $l_\infty(\alpha)$ must be at least $2n-3$. But not every set of $2n-3$ additional edges results in the increase of $l_n(g_n)$. The edges must be placed in such a way that the last $n-2$ of them actually do make length start to go back up after the first $n-2$ decreased length. In order to do this, they all must terminate sufficiently far away from v_0. Such a concern seems to force the penalty weight of such an element to be at least $n-3$ for a total length of $4n-9$ with respect to X_n. This heuristic argument leads to the following,

Conjecture 3.1: For $n \geq 2$, the depth of Thompson's Group F with respect to a generating set $X_n = \{x_0, x_1, \ldots, x_n\}$ is greater than or equal to $4n-9$.

3.3. **Penalty trees in $g_n\alpha$.** In order to analyze the effect on length of multiplying g_n by an element α of length at most $2n-8$, we must construct penalty trees in $D(g_n\alpha)$ that, if not minimal, are at least close enough to minimal to allow us to show that length does not increase. We may have to apply this construction to multiplication by elements that have length greater than $2n-8$ with respect to X_n, but those elements will always have length at most $2n-8$ with respect to X_∞, and moreover multiplying g_n by them will have the effect of modifying only the active portion of $D(g_n)$.

So, consider an element $\alpha \in F$ of length at most $2n-8$ with respect to X_∞ such that multiplication of g_n by α involves changing only the active portion of $D(g_n)$. Let $\alpha = \alpha_+\alpha_-$ with α_+ and α_- the positive and negative portions respectively of the normal form of g_n. Suppose that α_+ contains k generators and α_- contains l generators. Since $l_\infty(\alpha) \leq 2n-8$, the effect on the diagram of multiplication of g_n by α is the removal of k lower edges followed by the addition of l new lower edges with $k, l \leq 2n-8$. We now construct a penalty tree for $g_n\alpha$. We remark that the set of penalty type vertices of $g_n\alpha$ is exactly the set of penalty vertices of g_n.

Definition 3.1. *For element $\alpha \in F$ as above, construct a penalty tree $\widetilde{T}$ in $g_n\alpha$ by adding edges together with their endpoints as follows.*

1. *$\widetilde{T}$ contains all edges in the bottom path of $g_n\alpha$.*
2. *Recursively add additional edges to $\widetilde{T}$ by following the following steps.*
   (a) *Scan the diagram right to left starting at the rightmost vertex incident to a non-central edge until an essential cut vertex of g_n, say w, is encountered that is not already contained in $\widetilde{T}$.*
   (b) *Add to $\widetilde{T}$ the lower edge e_1 originating farthest left and terminating at w. From the initial vertex of e_1 add the lower edge e_2 terminating at $i(e_1)$ and originating the farthest left. Continue adding edges originating farthest left until a vertex already in $\widetilde{T}$ is reached.*
   (c) *Repeat (a) and (b) until every essential cut vertex of g_n is contained in $\widetilde{T}$.*
3. *For any penalty type vertex v not yet in $\widetilde{T}$, add to $\widetilde{T}$ the non-lower (i.e upper or central) edge which terminates at v and originates at an essential cut vertex of g_n, if such an edge exists.*
4. *Any penalty type vertex v that is not yet in $\widetilde{T}$ is the third vertex in a principal section of $\widetilde{T}$. For such a vertex v, add the unique lower edge terminating at v.*

We postpone the technical proofs of the following four lemmas to Section 4.

Lemma 2. *Let $\alpha \in F$ have length at most $2n-8$ with respect to X_∞. Suppose that the normal form of α contains the same number, k, of positive and negative* *generators and that multiplication of g_n by α involves modifying only the active portion of $D(g_n)$. If vertex w of $g_n\alpha$ that is an essential cut vertex of g_n is not contained in the bottom path of $D(g_n\alpha)$, then for the first vertex b to the left of w through $\widetilde{T}$ that is also on the bottom path of $D(g_n\alpha)$, we have $d_{\widetilde{T}}(w,b) \le k$.*

Lemma 3. *Let $\alpha \in F$ have length at most $2n-8$ with respect to X_∞, and suppose that multiplication of g_n by α involves changing only the active portion of $D(g_n)$. If the normal form of α contains the same number, say k, of positive as negative generators then $P_n(\widetilde{T}) \le P_n(g_n)$*

We remark that, though we do not need it for the proofs that follow, Lemma 3 implies that, under its hypotheses, $l_n(g_n\alpha) \le l_n(g_n)$.

To determine the effect on length of multiplying by an element α of length at most $2n-4$ whose normal form does not contain an equal number of positive and negative generators, we must carefully analyze how much more difficult it is to reach penalty type vertices through $\widetilde{T}$ than it is through T_n. For this, we require the following definition.

Definition 3.2. *A vertex v is said to be d* levels deep *if there are exactly d bottom edges covering v.*

Lemma 4. *Let α be as in Lemma 2, and let w and b be vertices of $g_n\alpha$ such that:*

- *w is an essential cut vertex of g_n,*
- *w is covered by a bottom edge of $g_n\alpha$,*
- *b is the first vertex left of w along $\widetilde{T}$ that is on the bottom path of $g_n\alpha$, and*
- *w is at most j levels deep.*

Then using the penalty tree $\widetilde{T}$ from Definition 3.1, $d_{\widetilde{T}}(b,w) \le j$

Lemma 5. *Let $\alpha \in F$ with $l_\infty(\alpha) \le 2n-8$ and let $\beta^{-1} \in X_n$. Suppose further that multiplication of g_n by $\alpha\beta$ involves changing only the active potion of $D(g_n)$. Denote by $\widetilde{T_1}$ the penalty tree in $g_n\alpha$ constructed in Definition 3.1 and $\widetilde{T_2}$ the tree constructed in Definition 3.1 for $g_n\alpha\beta$. Multiplication of $g_n\alpha$ by β involves adding an edge e covering a bottom vertex b of $g_n\alpha$. If e does not also cover a vertex w that is an essential cut vertex of g_n and that is at least $n-3$ levels deep in $g_n\alpha$ then $P_n(\widetilde{T_2}) = P_n(\widetilde{T_1}) - 1$.*

We now use the above Lemmas, whose proofs are postponed to Section 4, to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\alpha \in F$ be an element with $l_n(\alpha) \le 2n-8$. By Lemma 1, we may assume that the normal form of α contains more negative than positive generators. Let k be the number of positive generators in the the normal form of α, and write $\alpha = \alpha_+\gamma\beta$, where α_+ is the positive portion of the normal form of α, γ is the length k prefix of the negative portion of the normal form of α and β is the remaining negative portion of the normal form. Write, $\beta = \beta_1\beta_2\cdots\beta_m$, with $\beta_i^{-1} \in X_n$. Thus, $2k+m = l_\infty(\alpha) \le l_n(\alpha) \le 2n-8$ and $\frac{m}{2} \le n-4-k$.

By the construction of g_n, multiplication of g_n by $\alpha_+\gamma$ involves changing only the essential portion of $D(g_n)$. Let $\widetilde{T}$ be the penalty tree for $g_n\alpha_+\gamma$ given by Definition 3.1. By Lemma 3, $P_n(\widetilde{T}) \le P_n(g_n)$. Now, in the diagram for $g_n\alpha_+\gamma$, no essential cut vertex of g_n is more than k levels deep. Thus, in the diagram for $g_n\alpha_+\gamma\beta_1\beta_1\cdots\beta_i$, no essential cut vertex of g_n is more than $k+i$ levels deep. Since $\frac{m}{2} \leq n-4-k$, by Lemma 5 penalty tree $\widetilde{T_i}$ for $g_n\alpha_+\gamma\beta_1\beta_2\cdots\beta_i$ satisfies $P_n(\widetilde{T_i}) = P_n(T_{i-1}) - 1$. Therefore,

$$P_n(\widetilde{T_{\lceil \frac{m}{2} \rceil}}) = P_n(\widetilde{T}) - \left\lceil \frac{m}{2} \right\rceil \leq P_n(g_n) - \left\lceil \frac{m}{2} \right\rceil.$$

Since $l_\infty(g_n\alpha_+\gamma\beta_1\beta_2\cdots\beta_{\lceil \frac{m}{2} \rceil}) = l_\infty(g_n) + \lceil \frac{m}{2} \rceil$, we therefore have,

$$l_n(g_n\alpha_+\gamma\beta_1\beta_2\cdots\beta_{\lceil \frac{m}{2} \rceil}) \leq l_n(g_n) - \left\lceil \frac{m}{2} \right\rceil.$$

This implies that,

$$l_n(g_n\alpha) = l_n(g_n\alpha_+\gamma\beta) \leq l_n(g_n) - \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \leq l_n(g_n),$$

proving that g_n is a dead end of depth at least $2n-7$ with respect to the generating set X_n. Therefore, the dead end depth of F with respect to X_n is at least $2n-7$. □

4. Proofs

Proof of Lemma 2. Let w and b be as in the statement of the lemma. Let p be the path in $\widetilde{T}$ from b to w. Let $\alpha = \alpha_+\alpha_-$ be the normal form expression for α with α_+ consisting of the positive generator sequence and α_- the negative generator sequence. We claim that every vertex of p is on the bottom path of $g_n\alpha_+$. This is certainly true for w. Suppose towards a contradiction that there is a vertex in p not on the bottom path of $g_n\alpha$. Let w' be the first such vertex along p going from w to b. This means that w' is covered by a lower edge e of $g_n\alpha_+$. Now w' must have been reached by traveling along an edge in $g_n\alpha = g_n\alpha_+\alpha_-$ reaching farthest to the left from its terminal vertex, which by assumption lies on the bottom path of $g_n\alpha_+$. This is impossible, because w' is covered by the edge e of $g_n\alpha_+$. This contradiction establishes the claim that every vertex in p is on the bottom path of $g_n\alpha_+$.

Now, if the active portion of the bottom path g_n contains N vertices, then $g_n\alpha_+$ contains $N+k$ vertices in the active portion of its bottom path since α_+ has the effect of removing k edges from g_n. On the other hand, $g_n\alpha$ contains N vertices on the active portion of its bottom path, so there are at most k vertices that are exposed in $g_n\alpha_+$ but not exposed in $g_n\alpha$. Since no vertex along p except b is exposed in $g_n\alpha$, the length of p is at most k. □

Proof of Lemma 3. Since the tree $\widetilde{T}$ constructed in Definition 3.1 contains every penalty type vertex of $g_n\alpha$, it is a penalty tree for $g_n\alpha$. Working backwards through the construction of $\widetilde{T}$, we see that a vertex added in step 3 or 4 is a leaf of $\widetilde{T}$, so is not weighted. By Lemma 2, any vertex v added in step 2 is at most distance k through $\widetilde{T}$ to any essential cut vertex, w of g_n to its right. Such vertices w are the only possible leaves to the right of v through $\widetilde{T}$. Since $2k \leq 2n-8$ such vertices are at a distance less than $n-1$ from leaves through $\widetilde{T}$ and therefore not weighted. Thus, the only weighted vertices of $\widetilde{T}$ in the active portion of g_n are those in the essential portion of the bottom path of g_n. Since only the active portion of g_n is modified by multiplication by α, no vertex on the bottom path of g_n that is not weighted in T_n can become weighted in $\widetilde{T}$. Since the positive and negative portions of α are the same length, αg_n and $g_n\alpha$ have the same number of vertices on the essential portion of their bottom paths. Thus, $P_n(\widetilde{T}) \leq P_n(T_n) = P_n(g_n)$.

□

Proof of Lemma 4. Let w, b and j be as in the statement of the lemma. Every edge of the path p in $\widetilde{T}$ from w to b terminates either at a vertex on the bottom path of $g_n\alpha$ or a vertex that is the initial vertex of a lower edge terminating to the right of w and thus covering w. Therefore, $d_{\widetilde{T}}(w,b) \leq j$. □

Proof of Lemma 5. Let α and β be as in the statement of the lemma. By the construction of g_n, α cannot expose any upper edge of g_n. Thus, β indeed adds an edge e to the diagram of $g_n\alpha$. Let b be the bottom vertex of the diagram of $g_n\alpha$ that e covers. Suppose that every essential cut vertex of g_n that e covers, it does so to a depth of less than $n-3$. Let e_1 be the edge originating at b and terminating at $t(e)$. By abuse of notation, we may think of $\widetilde{T_1}$ as being a subtree of $D(g_n\alpha\beta)$. Penalty tree $\widetilde{T_2}$ is related to $\widetilde{T_1}$ by $\widetilde{T_2} = (\widetilde{T_1} \setminus \{e_1\}) \cup \{e\}$. Now, b is not among the last $n-2$ bottom vertices in the essential portion of $g_n\alpha$. Therefore, b is a weighted penalty vertex in $\widetilde{T_1}$. By Lemma 4, no essential cut vertex of g_n covered by e is at a distance of more than $n-3$ from b through $\widetilde{T_1}$. Thus, in the tree $\widetilde{T_2}$, b is not a weighted vertex. Since the remaining weighted vertices of $\widetilde{T_2}$ are the same as those of $\widetilde{T_1}$, we have $P_n(\widetilde{T_2}) = P_n(\widetilde{T_1})$.

□

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Undergraduate Mathematics Major, MSCS Department, University of Wisconsin–Stout, Menomonie, WI 54751

E-mail address: `halversonju@my.uwstout.edu`

Sponsor: Matthew Horak, Associate Professor, MSCS Department, University of Wisconsin–Stout, Menomonie, WI 54751

E-mail address: horakm@uwstout.edu