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A LIE ALGEBRA OF INTEGRALS FOR KEPLERIAN MOTION RESTRICTED TO THE PLANE

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Abstract. In this paper we consider the slightly simpler problem of Keplerian motion restricted to the plane rather than Keplerian motion in three dimensions as done in [3, page 11]. We parallel the three-dimensional problem in that we use the same actions to find invariants (integrals) but rather than working in a six-dimensional phase space to find six independent integrals we restrict ourselves to a four-dimensional phase space. In doing this, we find that we have three independent integrals and thus we have a three-dimensional Lie algebra.

1. Introduction

The two-body problem is classified as the behavior of two objects to attract each other according to the inverse square law [6, page 182]. It was Newton in 1685 who proposed that the gravitational force between two objects could be described by

$$F_G \propto \frac{mM}{r^2},$$

where m and M are the masses of the objects and r is their separation [4, page 149] [1]

Over 60 years before Newton proposed the inverse square law, Kepler had formulated his Laws of Planetary Motion which described the motion

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[1] To be historically correct, Newton was not the first to propose the inverse square law. In the early 1660's, Hooke attempted, without success, to experimentally show the inverse square relationship. For more history refer to [4, page 147].

of planets around the Sun. Kepler's Laws can be summarized as follows [4, page 166]:

1. The planets follow elliptical orbits around the Sun which is situated at a focus.
2. A line drawn from the sun to the planet will sweep out an equal area in an equal time.
3. The square of the period is proportional to the cube of the distance to the planet.

Newton realized that Kepler's first and third laws were a result of the inverse square law of attraction, and that Kepler's second law was a result of the conservation of angular momentum. In the first chapter of [3] it is shown that Kepler's second law is a result of an $O(3)$ symmetry, where $O(3)$ is the group of orthogonal 3×3 matrices. It is also shown that Keplerian elliptical motion in three dimensions has an $O(4)$ symmetry.

In this paper we construct the Lie algebra for Keplerian motion in the plane. In section 2 we define Lie algebras, Poisson brackets and actions on phase space. Section 2 concludes with the Lie algebra and action of $O(3)$. In section 3 we define the concept of an integral. In section 4 we review the results of Guillemin and Sternberg that closed orbits under the inverse square law have an $O(4)$ symmetry. Section 5 concludes the paper with the main result that the span of L_3, E_1, E_2 form a Lie algebra of integrals on the phase space of $\mathbb{R}^2$.

2. The Lie algebra and Action of $O(3)$

A Lie Algebra is a vector space $\mathcal{A}$ which is closed under a bracket operation $[\cdot,\cdot]$, where $[\cdot,\cdot]$ satisfies

1. $[a,b]$ is bilinear in a and b
2. $[\cdot,\cdot]$ is antisymetric, that is $[a,b] = -[a,b]$
3. Jacobi's Identity: $[a,[b,c]] = [[a,b],c] + [a,[b,c]]$

for all $a,b,c \in \mathcal{A}$ [3, page 2]. For example, if $m,p \in M_n(\mathbb{R})$, then

$$[m,p] = mp - pm$$

is the commutator bracket of m with p. If $\mathcal{M} \subseteq M_n(\mathbb{R})$ is closed under the commutator bracket then $\mathcal{M}$ is called a matrix Lie algebra. For example, let's consider the group $L_n(\mathbb{R})$ of invertible $n \times n$ matrices. If $\xi(t)$ is a one parameter subgroup of $L_n(\mathbb{R})$, then $\dot{\xi}(0)$ is an *infinitesimal* generator of $\xi(t)$. The set of generators $\mathcal{L}$, is closed under the commutator bracket and is therefore a Lie algebra. Moreover, $\mathcal{L}$ is called the Lie algebra of $L_n(\mathbb{R})$. See [5, Chapter 2] for more details.

For $f, g \in \mathcal{C} \subseteq C^\infty(\mathbb{R}^{2n})$ we define a new bracket, the Poisson bracket, of f with g to be

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right).$$

The Poisson bracket is an operation on $C^\infty(\mathbb{R}^{2n})$ with the same properties as the commutator bracket. That is, if a set is closed under the Poisson bracket then the set is a Lie algebra. In particular, the Poisson bracket has the following properties [3, page 8]:

1. $\{f, g\}$ is bilinear in f and g
2. $\{\cdot, \cdot\}$ is antisymmetric, that is $\{f, g\} = -\{g, f\}$
3. Jacobi's identity: $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$
4. Product Rule: $\{f, gh\} = \{f, g\}h + \{f, h\}g$

for all $f, g, h \in \mathcal{C}$. Given a Lie algebra $\mathcal{A}$, we say that $\sigma : \mathcal{A} \to C^\infty(\mathbb{R}^{2n})$ is an action on $\mathcal{A}$ if

$$\sigma[a, b] = \{\sigma(a), \sigma(b)\}.$$

for all $a, b \in \mathcal{A}$.

For example, let us construct the Lie algebra and action on the phase space for $O(3)$, the group of all possible rotations and reflections of the sphere. Rotations about the x_1, x_2, x_3 axes through angles θ, ϕ, ψ respectively are one-parameter subgroups of $O(3)$ [5, page 7]. In matrix form, these are

$$R_{x_1}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix},$$

$$R_{x_2}(\phi) = \begin{pmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{pmatrix},$$

$$R_{x_3}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The infinitesimal generators of $O(3)$ are [5, page 21],

$$\mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathcal{K} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the commutator bracket to $\mathcal{I}, \mathcal{J}, \mathcal{K}$ yields

$$[\mathcal{I}, \mathcal{J}] = \mathcal{K}$$
$$[\mathcal{J}, \mathcal{K}] = \mathcal{I}$$
$$[\mathcal{K}, \mathcal{I}] = \mathcal{J}.$$

Obviously the generators are closed under the commutator bracket and thus the Lie algebra of $O(3)$ is given by

$$o(3) = \{\alpha\mathcal{I} + \beta\mathcal{J} + \gamma\mathcal{K} \mid \alpha, \beta, \gamma \in \mathbb{R}^3\}.$$

Notice that $o(3)$ is isomorphic to $\mathbb{R}^3$ under the cross product. In particular we can identify $\mathcal{I}, \mathcal{J}, \mathcal{K}$ with the standard basis vectors, **i, j, k** of $\mathbb{R}^3$, respectively. Since we can identify $o(3)$ with $\mathbb{R}^3$, an action on $o(3)$ can be written as,

$$L(\xi) = \mathbf{L} \cdot \xi \tag{1}$$

where $\mathbf{L}$ is angular momentum and $\xi \in \mathbb{R}^3$. If $\mathbf{x} \in \mathbb{R}^3$ is the position and $\mathbf{p} \in \mathbb{R}^3$ is the momentum then

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}.$$

If for some $\mathcal{H} \in A \subseteq C^\infty(\mathbb{R}^{2n})$

$$\{\mathcal{H}, L(\xi)\} = 0,$$

then $L(\xi)$ is said to be invariant under $\mathcal{H}$. It will be shown in section 3 that $L(\xi)$ invariant under $\mathcal{H}$ implies $L(\xi)$ is constant and hence we will have a conservation of angular momentum. By the definition of angular momentum,

$$\mathbf{L} \cdot \mathbf{x} = 0,$$

which implies motion is in a plane. Conservation of angular momentum along with $\mathbf{L} \cdot \mathbf{x} = 0$ implies equal area swept out in equal times, Kepler's second law [3, page 10].

3. Integrals

To show Keplerian motion has $O(4)$ symmetry, we must first introduce the concept of an integral. For $\mathcal{H} \in \mathrm{A} \subseteq C^{\infty}(\mathbb{R}^{2n})$ and $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$ we define

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial \mathcal{H}}{\partial x_i} \end{aligned} \tag{2}$$

to be a Hamiltonian system with Hamiltonian $\mathcal{H}$ [5, page 41]. If $F(\mathbf{x}, \mathbf{p}) \in$ A is constant along the trajectories of (2) we say that $F(\mathbf{x}, \mathbf{p})$ is an integral. The following proposition relates the Hamiltonian $\mathcal{H}(\mathbf{x,p})$ to an integral via the Poisson bracket.

Proposition 1. *$F(\mathbf{x}, \mathbf{p})$ is an integral iff $\{F, \mathcal{H}\} = 0$.*

Proof. From the definition of an integral we know that F is constant and hence

$$\frac{dF}{dt} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial F}{\partial p_i} \frac{dp_i}{dt} \right) = 0. \tag{3}$$

From the definition of Poisson Bracket

$$\{F, \mathcal{H}\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial \mathcal{H}}{\partial x_i} \right). \tag{4}$$

Because $\mathcal{H}(\mathbf{x}, \mathbf{p})$ is a Hamiltonian we know that (2) holds. Substituting (2) into (4) yields

$$\frac{dF}{dt} = \{F, \mathcal{H}\} = 0$$

and hence F is an integral. Similarly, we can show the converse. □

Our next proposition shows that the set of integrals of a Hamiltonian are closed under the Poisson bracket and thus form a Lie algebra.

Proposition 2. *If g_i and g_k are integrals of a Hamiltonian system with Hamiltonian $\mathcal{H}$ then $\{g_i, g_k\}$ is also an integral.*

Proof. The proof follows directly from the Jacobi identity in that,

$$\{\mathcal{H}, \{g_i, g_k\}\} = \{\{\mathcal{H}, g_i\}, g_k\} + \{\{\mathcal{H}, g_k\}, g_i\}.$$

Since $\{\mathcal{H}, g_i\} = 0$ and $\{\mathcal{H}, g_k\} = 0$ and g_i, g_k trivially commute with zero we get the desired result,

$$\{\mathcal{H}, \{g_i, g_k\}\} = 0.$$

which concludes the proof. □

Our last proposition shows that the set of integrals is closed under ordinary multiplication.

Proposition 3. *If g_i and $g_k \in A$ and $\mathcal{H} \in A$ such that g_i and g_k are integrals and $\mathcal{H}$ is a Hamiltonian then $g_i g_k$ is also an integral.*

Proof. Notice that $\{\mathcal{H}, g_i g_k\}$, using the product rule, reduces to

$$\{\mathcal{H}, g_i\}g_k + \{\mathcal{H}, g_k\}g_i.$$

Since g_i, g_k are integrals

$$\{\mathcal{H}, g_i g_k\} = 0$$

and thus $g_i g_k$ is also an integral. □

4. The Lie algebra and action of O(4)

In [3, pages 13-15], it is shown that Keplerian motion has an $O(4)$ symmetry. If we define $\mathbf{F}$, to be

$$\mathbf{F} = \mathbf{p} \times \mathbf{L} + \frac{\mathbf{x}}{r} \tag{5}$$

then

$$F(\xi) = \mathbf{F} \cdot \xi \tag{6}$$

is a map from $o(3) \sim \mathbb{R}^3 \to \mathbb{R}^{2n}$. Writing (5) as

$$\mathbf{F} = \mathbf{p} \times (\mathbf{x} \times \mathbf{p}) + \frac{\mathbf{x}}{r} \tag{7}$$

and expanding the triple cross product yields

$$\mathbf{F} = (\mathbf{p} \cdot \mathbf{p})\mathbf{x} - (\mathbf{x} \cdot \mathbf{p})\mathbf{p} + \frac{\mathbf{x}}{r}. \tag{8}$$

It is nontrivial to show that if $\xi, \eta \in o(4)$, then

$$\begin{aligned} \{L(\xi), L(\eta)\} &= L([\xi, \eta]) \\ \{L(\xi), F(\eta)\} &= F([\xi, \eta]) \\ \{F(\xi), F(\eta)\} &= -\mathcal{H}L([\xi, \eta]), \end{aligned} \tag{9}$$

where $\mathcal{H}$ is the Hamiltonian of Keplerian motion and is given by

$$\mathcal{H} = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{1}{r}$$

(see [3, pages 13-15] for details)[2]. Thus, $\mathcal{H}$ describes a conservation of energy where

$$\text{Potential} \propto \frac{1}{r}. \tag{10}$$

It follows directly from (10) that

$$\text{Force} \propto \frac{1}{r^2}.$$

However, it is obvious that (9) is not closed under the Poisson bracket. Thus we define

$$\mathbf{E} = \phi \mathbf{F}, \tag{11}$$

where $\phi = \frac{1}{\sqrt{-2\mathcal{H}}}$. Hence we get the new set of relations

$$\begin{aligned} \{L(\xi), L(\eta)\} = &\quad L([\xi, \eta]) \\ \{L(\xi), E(\eta)\} = &\quad E([\xi, \eta]) \\ \{E(\xi), E(\eta)\} = &\quad L([\xi, \eta]), \end{aligned} \tag{12}$$

where $E(\eta) = \mathbf{E} \cdot \eta$. For $\xi, \eta \in g$ and $\mathcal{H} < 0$ the action on $o(4)$ is given by

$$\sigma(\xi, \eta) = \mathbf{L} \cdot \xi + \mathbf{E} \cdot \eta. \tag{13}$$

The action of $o(4)$ is the vector space which spans $L_1, L_2, L_3, E_1, E_2, E_3$. That is the action of $o(4)$ describes the portion of a six-dimensional phase space where $\mathcal{H} < 0$.

With an understanding of $o(3)$ and $o(4)$ we now move to the main result of this paper, that of constructing a Lie algebra of integrals for Keplerian motions on the phase space of $\mathbb{R}^2$. In this two-dimensional setting we will not get a six-dimensional algebra and hence we not have an $O(4)$ symmetry. Instead we will get a three-dimensional Lie algebra isomorphic to o(3).

5. Keplerian Motion in $\mathbb{R}^2$

In this section we construct a Lie algebra of integrals for the Hamiltonian system with Hamiltonian $\mathcal{H} = \frac{1}{2}\|\mathbf{p}^2\| + \frac{1}{r}$ such that $\mathbf{p}, \mathbf{r} \in \mathbb{R}^2$. That is, we restrict Keplerian motion to the xy-plane. If we let L_3 be the third component of $\mathbf{L}$, it follows that

$$L_3 = L(\mathbf{k}) = x_1 p_2 - x_2 p_1. \tag{14}$$

[2]In actuality, the Hamiltonian for Keplerian motion is $\mathcal{H} = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{k}{r}$, where k is a constant (see [3, page 15] for details). For simplicity, we let k=1.

Also we let E_1, E_2 be the first two components of $\mathbf{E}$ such that

$$E_1 = \phi\left(\zeta_1 + \frac{x_1}{r}\right) \qquad (15)$$
$$E_2 = \phi\left(\zeta_2 + \frac{x_2}{r}\right)$$

where ζ_1, ζ_2 are the first and second components of $\mathbf{p} \times \mathbf{L}$, respectively and $\phi = \frac{1}{\sqrt{-2\mathcal{H}}}$. We now show that L_3, E_1, E_2 are closed under the Poisson bracket and that they commute with the Hamiltonian.

First we show that L_3, E_1, E_2 are integrals. Beginning with L_3, we write the expression

$$\{\mathcal{H}, L_3\} = \left\{\frac{1}{2}\|\mathbf{p}\|^2 + \frac{1}{r},\ x_1p_2 - x_2p_1\right\}. \qquad (16)$$

The linearity of the Poisson bracket and the product rule reduce (16) to

$$\{\mathcal{H}, L_3\} = \left\{\frac{1}{2}\|\mathbf{p}\|^2, x_1\right\} p_2 - \left\{\frac{1}{2}\|\mathbf{p}\|^2, x_2\right\} p_1 + \left\{\frac{1}{r}, p_2\right\} x_1 - \left\{\frac{1}{r}, p_1\right\} x_2. \qquad (17)$$

Evaluating (17) yields

$$\{\mathcal{H}, L_3\} = p_1p_2 - p_2p_1 - \frac{x_2x_1}{r^3} + \frac{x_1x_2}{r^3} = 0. \qquad (18)$$

Before showing E_1, E_2 are integrals we notice that if we define

$$\frac{\partial f}{\partial \mathbf{x}} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \qquad (19)$$

and

$$\frac{\partial f}{\partial \mathbf{p}} = \left\langle \frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, \ldots, \frac{\partial f}{\partial p_n} \right\rangle, \qquad (20)$$

then the Poisson bracket of f with g can be expressed as [2, 1]

$$\{f, g\} = \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{x}}. \qquad (21)$$

Moreover notice that

$$\mathbf{F} \cdot \alpha = (\mathbf{p} \cdot \mathbf{p})(\mathbf{x} \cdot \alpha) - (\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \alpha) + \frac{\mathbf{x} \cdot \alpha}{r}, \qquad (22)$$

which reduces to

$$\mathbf{F} \cdot \alpha = \hat{\mathcal{H}}(\mathbf{x} \cdot \alpha) - (\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \alpha) \qquad (23)$$

where $\hat{\mathcal{H}} = \|\mathbf{p}\|^2 + \frac{1}{r}$. It can be shown that

$$\frac{\partial(\mathbf{F} \cdot \alpha)}{\partial \mathbf{x}} = \frac{\partial \hat{\mathcal{H}}}{\partial \mathbf{x}}(\mathbf{x} \cdot \alpha) + \hat{\mathcal{H}}\alpha - \mathbf{p}(\mathbf{p} \cdot \alpha) \qquad (24)$$

and

$$\frac{\partial(\mathbf{F}\cdot\alpha)}{\partial\mathbf{p}} = \frac{\partial\hat{\mathcal{H}}}{\partial\mathbf{p}}(\mathbf{x}\cdot\alpha) - \mathbf{x}(\mathbf{p}\cdot\alpha) - (\mathbf{x}\cdot\mathbf{p})\alpha. \tag{25}$$

Notice also that

$$\mathbf{p}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{p}} = \|\mathbf{p}\|^2 \tag{26}$$

and

$$\mathbf{x}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{x}} = -\frac{1}{r}. \tag{27}$$

Using (24) and (25) the Poisson bracket of $\mathbf{F}\cdot\alpha$ with $\mathcal{H}$ is given by

$$\begin{aligned}\{\mathbf{F}\cdot\alpha,\mathcal{H}\} = {} & \frac{\partial\hat{\mathcal{H}}}{\partial\mathbf{x}}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{p}}(\mathbf{x}\cdot\alpha) + \hat{\mathcal{H}}\frac{\partial\mathcal{H}}{\partial\mathbf{p}}\cdot\alpha \\ & - \mathbf{p}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{p}}(\mathbf{p}\cdot\alpha) - \frac{\partial\hat{\mathcal{H}}}{\partial\mathbf{p}}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{x}}(\mathbf{x}\cdot\alpha) \\ & + \mathbf{x}\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{x}}(\mathbf{p}\cdot\alpha) + \alpha\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{x}}(\mathbf{x}\cdot\mathbf{p}).\end{aligned} \tag{28}$$

Substituting (26), (27) into (28) and simplifying reduces (28) to

$$\{\mathbf{F}\cdot\alpha,\mathcal{H}\} = \{\hat{\mathcal{H}},\mathcal{H}\}(\mathbf{x}\cdot\alpha) + \alpha\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{x}}(\mathbf{x}\cdot\mathbf{p}). \tag{29}$$

Substituting in the expression for $\hat{H}$ and using the Product Rule, (29) reduces to

$$\{\mathbf{F}\cdot\alpha,\mathcal{H}\} = \left\{\frac{1}{2}\|\mathbf{p}\|^2,\mathcal{H}\right\}(\mathbf{x}\cdot\alpha) + \alpha\cdot\frac{\partial\mathcal{H}}{\partial\mathbf{x}}(\mathbf{x}\cdot\mathbf{p}). \tag{30}$$

Using the definition of the Poisson bracket and

$$\frac{\partial\mathcal{H}}{\partial\mathbf{x}} = \frac{\partial\hat{\mathcal{H}}}{\partial\mathbf{x}} = \frac{\mathbf{x}}{r^3}, \tag{31}$$

we can write (30) as

$$\{\mathbf{F}\cdot\alpha,\mathcal{H}\} = -\frac{(\mathbf{x}\cdot\mathbf{p})(\mathbf{x}\cdot\alpha)}{r^3} + \frac{(\mathbf{x}\cdot\mathbf{p})(\mathbf{x}\cdot\alpha)}{r^3} = 0. \tag{32}$$

Hence, F_1, F_2 are integrals. Since ϕ is trivially an integral, Proposition 3 says that E_1, E_2 are also integrals.

To show closure we begin with

$$\{L_3, E_1\} = \left\{L_3, \left(\zeta_1 + \frac{x_1}{r}\right)\phi\right\}. \tag{33}$$

Using the linearity property of the Poisson bracket and the product rule we can rewrite (33) as

$$\{L_3, E_1\} = \{L_3,\zeta_1\}\phi\{L_3,\phi\}\zeta_1\left\{L_3,\frac{x_1}{r}\right\}\phi + \{L_3,\phi\}\frac{x_1}{r}. \tag{34}$$

It can be shown for any integral L and some function of $\mathcal{H}$, $g(\mathcal{H})$ that

$$\{L, g(\mathcal{H})\} = 0. \tag{35}$$

It can also be shown that

$$\begin{aligned} \{L_3, \zeta_1\} &= \zeta_2 \\ \{L_3, \zeta_2\} &= -\zeta_1 \\ \{L_3, x_1\} &= x_2 \\ \{L_3, x_2\} &= -x_1. \end{aligned} \tag{36}$$

Using (35) and (36) reduces (34) to

$$\{L_3, E_1\} = \phi\left(\zeta_2 + \frac{x_2}{r}\right) = E_2. \tag{37}$$

In a similar manner,

$$\{L_3, E_2\} = \phi\left(-\zeta_1 - \frac{x_1}{r}\right) = -E_1. \tag{38}$$

To show $\{E_1, E_2\} = L_3$ we return to the Poisson bracket defined in (21). Using (23) we can write

$$\{\mathbf{F}\cdot\alpha, \mathbf{F}\cdot\beta\} = \{\hat{\mathcal{H}}(\mathbf{x}\cdot\alpha) - (\mathbf{x}\cdot\mathbf{p})(\mathbf{p}\cdot\alpha), \hat{\mathcal{H}}(\mathbf{x}\cdot\beta) - (\mathbf{x}\cdot\mathbf{p})(\mathbf{p}\cdot\beta)\}. \tag{39}$$

Using the linearity of the Poisson bracket (39) yields

$$\begin{aligned} \{\mathbf{F}\cdot\alpha, \mathbf{F}\cdot\beta\} = &\overbrace{\{\hat{\mathcal{H}}(\mathbf{x}\cdot\alpha), \hat{\mathcal{H}}(\mathbf{x}\cdot\beta)\}}^{(a)} \\ &+ \overbrace{\{\hat{\mathcal{H}}(\mathbf{x}\cdot\alpha), -(\mathbf{x}\cdot\mathbf{p})(\mathbf{p}\cdot\beta)\}}^{(b)} \\ &+ \overbrace{\{-(\mathbf{x}\cdot\mathbf{p})(\mathbf{p}\cdot\alpha), \hat{\mathcal{H}}(\mathbf{x}\cdot\beta)\}}^{(c)} \\ &+ \overbrace{\{-(\mathbf{x}\cdot\mathbf{p})(\mathbf{p}\cdot\alpha), -(\mathbf{x}\cdot\mathbf{p})(\mathbf{p}\cdot\beta)\}}^{(d)}. \end{aligned} \tag{40}$$

Notice that

$$\mathbf{x}\cdot\frac{\partial\hat{\mathcal{H}}}{\partial\mathbf{x}} = -\frac{1}{r} \tag{41}$$

and

$$\frac{\partial\hat{\mathcal{H}}}{\partial\mathbf{p}} = 2\mathbf{p}. \tag{42}$$

Using (41) and (42), (a) becomes

$$\{\hat{\mathcal{H}}(\mathbf{x}\cdot\alpha), \hat{\mathcal{H}}(\mathbf{x}\cdot\beta)\} = -2\hat{\mathcal{H}}[(\mathbf{x}\cdot\alpha)(\mathbf{p}\cdot\beta) - (\mathbf{x}\cdot\beta)(\mathbf{p}\cdot\alpha)]. \tag{43}$$

Computing (b) is similar to (a), but we will need

$$p \cdot \frac{\partial \hat{\mathcal{H}}}{\partial \mathbf{p}} = 2\|\mathbf{p}\|^2 \tag{44}$$

and

$$\hat{\mathcal{H}}(\alpha \cdot \beta)(\mathbf{x} \cdot \mathbf{p}) = 0. \tag{45}$$

The end result of (b) is

$$\{\hat{\mathcal{H}}(\mathbf{x} \cdot \alpha), -(\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \beta)\} = \|\mathbf{p}\|^2(\mathbf{x} \cdot \alpha)(\mathbf{p} \cdot \beta) - \beta \cdot \frac{\partial \hat{\mathcal{H}}}{\partial \mathbf{x}}(\mathbf{x} \cdot \alpha)(\mathbf{x} \cdot \mathbf{p}). \tag{46}$$

In a similar fashion, (c) reduces to

$$\begin{aligned} &\{-(\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \alpha), \hat{\mathcal{H}}(\mathbf{x} \cdot \beta)\} \\ &\qquad = -\|\mathbf{p}\|^2(\mathbf{p} \cdot \alpha)(\mathbf{x} \cdot \beta) - \alpha \cdot \frac{\partial \hat{\mathcal{H}}}{\partial \mathbf{x}}(\mathbf{x} \cdot \beta)(\mathbf{x} \cdot \mathbf{p}). \end{aligned} \tag{47}$$

Finally, using strictly the Poisson bracket and the distributive law (d) reduces to

$$\begin{aligned} &\{-(\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \alpha), -(\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \beta)\} \\ &\quad = (\mathbf{p} \cdot \mathbf{x})(\mathbf{p} \cdot \alpha)(\mathbf{p} \cdot \beta) + (\mathbf{p} \cdot \beta)(\mathbf{p} \cdot \alpha)(\mathbf{x} \cdot \mathbf{p}) \\ &\qquad - (\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \alpha)(\mathbf{p} \cdot \beta) - (\mathbf{p} \cdot \alpha)(\mathbf{x} \cdot \mathbf{p})(\mathbf{p} \cdot \beta) = 0. \end{aligned} \tag{48}$$

Adding (46) and (47), factoring out $\|\mathbf{p}\|^2$, and substituting (31) (which will help cancel two terms), we are left with

$$(b) + (c) = \|\mathbf{p}\|^2[(\mathbf{x} \cdot \alpha)(\mathbf{p} \cdot \beta) - (\mathbf{x} \cdot \beta)(\mathbf{p} \cdot \alpha)]. \tag{49}$$

Finally, adding (49) and (43) and using the definition of $\hat{\mathcal{H}}$ we get

$$\{\mathbf{F} \cdot \alpha, \mathbf{F} \cdot \beta\} = -2\mathcal{H}[(\mathbf{x} \cdot \alpha)(\mathbf{p} \cdot \beta) - (\mathbf{x} \cdot \beta)(\mathbf{p} \cdot \alpha)]. \tag{50}$$

If $\alpha = \mathbf{i}$ and $\beta = \mathbf{j}$, then

$$\{F_1, F_2\} = -2\mathcal{H}L_3. \tag{51}$$

By definition,

$$\{E_1, E_2\} = \{\phi F_1, \phi F_2\}. \tag{52}$$

Moreover, it can be shown that

$$\{E_1, E_2\} = \phi^2\{F_1, F_2\} = L_3, \tag{53}$$

since $\phi = \frac{1}{\sqrt{-2\mathcal{H}}}$.

We have shown that E_1, E_2, L_3 commute with

$$\mathcal{H} = \frac{1}{2}\|\mathbf{p}\|^2 + \frac{1}{r},$$

the Hamiltonian of Keplerian elliptical motion and hence are integrals. We have also shown that E_1, E_2, L_3 are closed under the Poisson bracket and thus form a three-dimensional Lie algebra. Moreover, we leave it to the reader to show that the Lie algebra generated by E_1, E_2, L_3 is isomorphic to $O(3)$.

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