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RELATIVE GOLDBACH PARTITIONS AND GOLDBACH'S CONJECTURE

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Abstract. In this note, we utilize techniques from discrete mathematics to develop first an inequality, and then second a counting formula that is connected to Goldbach's conjecture. In order to do this, we introduce the notion of a Relative Goldbach Partition.

1. Introduction

The allure of Goldbach's conjecture has captured the interest of many. It has appeared in the story lines of television shows, movies, and recent novels such as Uncle Petros and Goldbach's Conjecture by Apostolos Doxiodis [1]. The conjecture is often introduced to undergraduates in their first or second year in discrete mathematics texts as in [2]. The conjecture says that every even integer greater than 2 can be written as the sum of two prime numbers. In this note, we connect the pigeon-hole principle to Goldbach's conjecture for certain even integers and then develop an interesting counting formula utilizing what we term Relative Goldbach Partitions. All of our results are accessible to students in a first course in discrete mathematics.

2. A Pigeon-Hole Argument

We denote the Euler totient of a positive integer n by $\phi(n)$. Recall that this gives the number of positive integers less than n that are relatively prime to n. We define $\rho(n)$ for a positive, even integer n to be the number of primes that are less than n but are not among the prime factors of n. We claim that if $\rho(n) > \phi(n)/2$, then n can be written as the sum of two distinct primes. To see this, observe that an even integer n can be written as the sum of two positive numbers in $n/2$ ways:

$$1 + n - 1, 2 + n - 2, \ldots, n/2 + n/2.$$

If the first summand is relatively prime to n, then so is the second. It follows that $\phi(n)/2$ of the sums consist of two numbers that are relatively prime to n. We think of $\rho(n)$ as pigeons and $\phi(n)/2$ as pigeon-holes. Thus if $\rho(n) > \phi(n)/2$, then n can be written as a sum of two distinct primes. As an example, we apply our pigeon-hole argument to the number 22. Since $\rho(22) = 6$ and $\phi(22)/2 = 5$, we are guaranteed that a sum of distinct primes exists. Note that 22 = 11 + 11, 3 + 19, and 5 + 17.

Unfortunately, the inequality $\rho(n) > \phi(n)/2$ fails more often than it holds. In Figure 1, we compare a graph of $\rho(n)$ with $\phi(n)/2$ for even values of n out to 1000.

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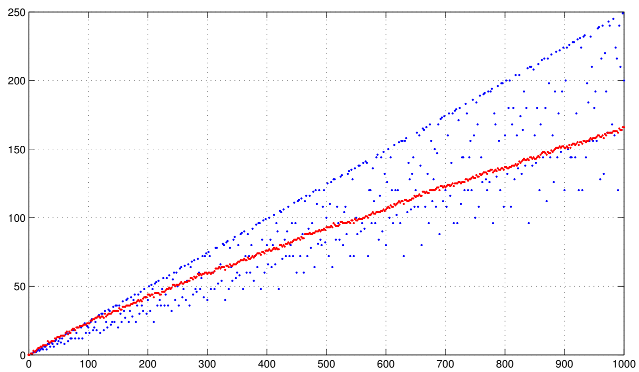

FIGURE 1. $\phi(n)/2$ in blue compared to $\rho(n)$ for even n out to 1000

The blue dots below the red line in Figure 1 represent values of n for which we may apply our pigeon-hole argument.

3. Relative Goldbach Partitions

A pair of prime numbers that sum to an even integer n is called a *Goldbach Partition*. If Goldbach's conjecture holds, then every even integer n greater than 2 has at least one Goldbach Partition. For any positive integer n, we define a *Relative Goldbach Partition* (RGP) to be a pair of positive numbers that sum to n such that the numbers are relatively prime to n and the summands are not themselves prime numbers. If $n-1$ is not prime, then n has at least one RGP namely $\{1, n-1\}$. It is interesting to note that the first number that has two RGP's is 34. The number 34 can be written as $1+33$ and $9+25$. For an integer n, we denote the number of unordered RGP's by $R(n)$. The sequence $R(n)$ is a recent addition to the online encyclopedia of integer sequences. It matches A185279 while $R(2n)$ is the same as A141095 (see [3] and [4]). We also denote the number of unordered Goldbach Partitions consisting of distinct primes by $G(n)$. In Figure 2, we see the graph of $R(n)$ and we invite the reader to compare it to the well known graph of Goldbach's comet.

4. A Counting Formula

We next derive a formula for $G(n)$ utilizing the functions $\rho(n)$, $R(n)$ and $\phi(n)$. We claim that for an even integer n,

$$G(n) = \rho(n) + R(n) - \phi(n)/2.$$

To see this, consider the $\phi(n)$ numbers that are relatively prime to n. Pairing these numbers into sums of n yields three possibilities. The sum contributes to the value of $G(n)$, the sum contributes to the value of $R(n)$, or one of the summands is prime and the other is a number that is relatively prime to n. We call this a *type 3* sum. There are two ways to count the number of type 3 sums. The first way is to take all of the $\phi(n)$ many numbers, subtract all primes that remain, and delete all pairs

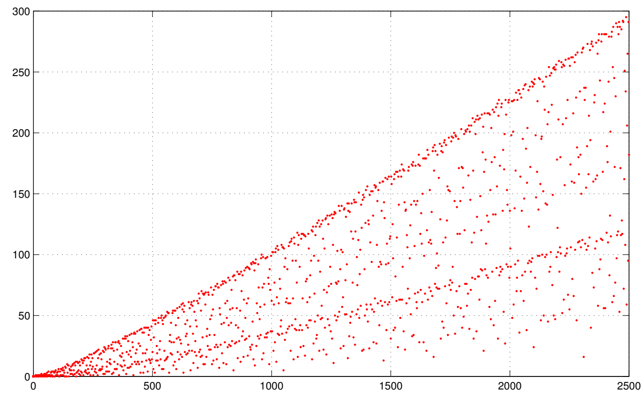

FIGURE 2. The comet corresponding to $R(n)$

that contribute to $R(n)$. Thus the number of type 3 sums is equal to

$$\phi(n) - \rho(n) - 2R(n).$$

The second way to count the number of type 3 sums is to do the following. Start only with the primes that contribute to the value of $\rho(n)$. If one subtracts all of the primes that are a member of a distinct Goldbach Partition, then what is left is again equal to the number of type 3 sums. This is the same as

$$\rho(n) - 2G(n).$$

Equating the two expressions that represent type 3 sums and solving for $G(n)$ gives the desired result.

Our counting formula implies that if $\rho(n) + R(n) > \phi(n)/2$, then $G(n) > 0$. The latter inequality appears to hold for all integers greater than 6. This means that one could rephrase Goldbach's conjecture to say that every even integer greater than 6 can be written as the sum of two *distinct* primes. Thus if Goldbach's conjecture is false, then there exists an integer n such that $\rho(n) + R(n) = \phi(n)/2$. Based on the computational efforts of many (see e.g. [5]), such an n would no doubt be very large!

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