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PROPERTIES OF THE ITERATES OF THE WEIERSTRASS-$\wp$ FUNCTION

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Abstract. This paper discusses several properties of the Weierstrass-$\wp$ function, as defined on the fundamental parallelogram $\mathbb{C}/\Gamma$, where $\mathbb{C}$ is the complex plane and Γ is the lattice generated by ω_1 and ω_2. Using the addition formula for $\wp(z_1 + z_2)$, we develop a reccurence relation for $\wp(nz)$ in terms of $\wp(z)$. We then examine the degree of this expression, some coefficients, and patterns concerning the poles of this function. We also consider the geometric interpretation of taking an arbitrary z_0 and adding it to itself, both in the fundamental parallelogram $\mathbb{C}/\Gamma$ and the elliptic curve generated by $\wp(z)$ and $\wp'(z)$.

1. Introduction

Elliptic functions are single-valued, doubly-periodtic meromorphic functions of a complex variable that are inverses of elliptic integrals. Elliptic functions satisfy many nonlinear differential equations arising in mathematical physics and, therefore, are useful in applications.

Every person who knows trionometry will agree that the periodicity property

$$\sin(x + 2k\pi) = \sin x \text{ and } \cos(x + 2k\pi) = \cos x, \forall x \in \mathbb{C}, k \in \mathbb{Z} \tag{1.1}$$

is of fundamental importance in all scientific calculations. An elliptic function f satisfies

$$f(z + m\omega_1 + n\omega_2) = f(z), \forall z \in \mathbb{C}, \forall m, n \in \mathbb{Z} \tag{1.2}$$

for a pair of primative periods ω_1 and ω_2. Thus (1.2) can be viewed as a generalization of (1.1). We also recall the angle-addition formulas

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1,$$
$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2,$$
$$\forall \theta_1, \theta_2 \in \mathbb{R}$$

Such formulas lead to great simplifications in calculations. Elliptic functions satisfy certain properties of a similar nature. But they involve derivatives, cf. (3.1) below.

There are two "simple" types of elliptic functions: Jacobian elliptic functions and Weierstrass elliptic functions, which are classified according to the order of poles. Properties of such elliptic functions may be found in [3].

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Let $\wp$ be a Weierstrass-$\wp$ elliptic function. In this paper, we are primarily interested in deriving the recurrence formula for $\wp(nz)$, for any $z \in \mathbb{C}$ and $n = 1, 2, ...$ Such a formula may be likened to the multiple-angle formula

$$\sin(n\theta) = \begin{cases} n\cos\theta[\sin\theta - \frac{(n^2-2^2)}{3!}\sin^3\theta + \frac{(n^2-2^2)(n^2-4^2)}{5!}\sin^5\theta \pm ...]; & \text{for } n \text{ even, } n > 0 \\ n\sin\theta - \frac{n(n^2-1^2}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta \pm ... & \text{for } n \text{ odd, } n > 0 \end{cases}$$

(Note that the above sums terminate after finitely many terms.) But we could not find the formula for $\wp(nz)$ in [3] or other references. This motivated us to write this article.

The orginization of the paper is as follows: In section 2, we provide some preliminary material. In section 3, we derive the recurrence formula for $\wp(nz)$ and some associated properties.

2. Preliminaries

2.1. **Chebyshev Polynomials.** Some nice properties of the Chebyshev Polynomials help us derive a recurrence relation for $\wp(nz)$ and certain associated properties of the resulting explicit rational function. A good reference for Chebyshev Polynomials may be found in [2].

Definition 2.1. The Chebyshev Polynomial of the first-kind, $T_n(x)$, is a polynomial in x of degree n defined by:

$$\cos(n\theta) = T_n(x) \text{ where } x = \cos\theta. \tag{2.1}$$

An example of a neat property of Chebyshev Polynomials is given by the following theorem.

Theorem 2.2. *The fundamental recurrence relation that generates all of the polynomials $T_n(x)$ is as follows:*

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \text{ for } n = 2, 3, ... \tag{2.2}$$

with initial conditions

$$T_0(x) = 1 \text{ and } T_1(x) = x. \tag{2.3}$$

2.2. **Weierstrass-$\wp$ Function.** This investigation is natural because the Weierstrass-$\wp$ Function shares some properties with the aforementioned trigonometric functions, such as periodicity odd/even characteristics. The Weierstrass-$\wp$ Function is given as follows. Additional information can be found in [1].

Definition 2.3. Let points $\omega_1, \omega_2 \in \mathbb{C}$ be non-colinear with 0 on the complex plane. Define the lattice Γ as all points of the form

$$\omega \in \Gamma \iff \omega = a\omega_1 + b\omega_2 \ \ \forall a, b \in \mathbb{Z}. \tag{2.4}$$

The Weierstrass-$\wp$ function is defined as:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Gamma,\omega\neq 0} \left(\frac{1}{z-\omega} - \frac{1}{\omega^2} \right), \ \ \forall z \in \mathbb{C}. \tag{2.5}$$

Remark 2.4. Note that $\wp(z)$ is even, and more importantly, that $\wp(z)$ is double periodic, with periods of ω_1 and ω_2, i.e.,

$$\wp(z + \omega_1) = \wp(z + \omega_2) = \wp(z) \ \ \forall z \in \mathbb{C}. \tag{2.6}$$

Because of this, we may examine $\wp(z)$ on a fundamental parallelogram $\mathbb{C}/\Gamma$ defined by ω_1 and ω_2. Note that $\wp(z)$ contains its only poles at lattice points, and these are *poles of order two.*

Theorem 2.5. *Using power series expansions and pole/zero comparisons, we have*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{2.7}$$

where

$$g_2 = 60 \sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^4} \text{ and } g_3 = 140 \sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^6}, \tag{2.8}$$

and

$$\wp'(z) = \frac{-2}{z^3} + 6 \sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^4} z + 20 \sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^6} z^3. \tag{2.9}$$

Remark 2.6. From here we see that $\wp'(z)$ is also double-periodic on the fundamental lattice lattice, and that it is odd. Because of this, we see that the half-periods must be zeroes of $\wp'(z)$. In our fundamental parallelorgram, we have

$$\wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1+\omega_2}{2}\right) = 0. \tag{2.10}$$

We also see that the points $(\wp(z), \wp'(z))$ lie on the curve $y^2 = 4x^3 - g_2x - g_3$. This is an elliptic curve with roots at $x = \wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$.

Using the relationship between $\wp(z)$ and this curve, we can now begin to derive a recurrence relation.

3. Recurrence Relation for $\wp(nz)$ and its Derivation

Theorem 3.1. *For $z_1, z_2 \in \mathbb{C}$,*

$$\wp(z_1+z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2. \tag{3.1}$$

If $z_1 \equiv z_2 \bmod \Gamma$, the limit as $z_2 \to z_1$ gives

$$\wp(2z_1) = -2\wp(z_1) + \frac{1}{4}\left(\frac{\wp''(z_1)}{\wp'(z_1)}\right)^2. \tag{3.2}$$

Its proof may be found in [1] and we omit it. Now we can give a recurrence relation for $\wp(nz)$ in the following.

Theorem 3.2. *If n is even, we have*

$$\wp(nz) = -2\wp(\frac{z}{2}) + \frac{1}{4}\left(\frac{\wp''(\frac{z}{2})}{\wp'(\frac{z}{2})}\right)^2, \tag{3.3}$$

and for n odd we have

$$\wp(z+(n-1)z) = -\wp(z) - \wp((n-1)z) + \frac{1}{4}\left(\frac{\wp'(z)-\wp'((n-1)z)}{\wp(z)-\wp((n-1)z)}\right)^2. \tag{3.4}$$

Note: its proof is available in [1].

Theorem 3.3. *We have*

$$\wp(nz) = P_n(\wp(z)) = \frac{1}{n^2} + R_n(\wp(z)), \tag{3.5}$$

where R_n is some rational function of $\wp(z)$ with degree less than 0, $\forall n \in \mathbb{Z}$

Proof. We write the power series expansion for $\wp(z)$ at the origin:

$$\wp(z) = \frac{1}{z^2} + 3\sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^4}z^2 + 5\sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^6}z^4 + ..., \tag{3.6}$$

so

$$\wp(nz) = \frac{1}{n^2z^2} + 3\sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^4}n^2z^2 + 5\sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^6}n^4z^4 + ... \tag{3.7}$$

Define

$$H_n(z) = 3\sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^4}n^2z^2 + 5\sum_{\omega\in\Gamma,\omega\neq 0} \frac{1}{\omega^6}n^4z^4 + ..., \tag{3.8}$$

thus

$$\wp(nz) = \frac{1}{n^2z^2} + H_n(z). \tag{3.9}$$

Note that as $z \to 0$, $H_n(z) \to 0$ as well. We also know that $\wp(nz) = P_n(\wp(z))$. Substitute in our expansion

$$\wp(nz) = P_n\left(\frac{1}{z^2} + H_1(z)\right). \tag{3.10}$$

Since $P_n(x)$ is some rational function of x, define $\tilde{P_n}(x)$ as the quotient with no term of degree less than 0, and $R_n(x)$ as the remainder of degree less than 0, we thus obtain

$$\begin{aligned}\wp(nz) = P_n\left(\frac{1}{z^2} + H_1(z)\right) &= \tilde{P_n}\left(\frac{1}{z^2} + H_1(z)\right) + R_n\left(\frac{1}{z^2} + H_1(z)\right)\\ &= \frac{1}{n^2z^2} + H_n(z).\end{aligned} \tag{3.11}$$

First note that as $z \to 0$, $(\frac{1}{z^2} + H_1(z)) \to \infty$. But since $\deg R_n < 0$, $R_n(\frac{1}{z^2} + H_1(z)) \to 0$, so R_n will not contribute to the $\frac{1}{z^2}$ term. As for the $\tilde{P_n}$ term,

$$\tilde{P_n}\left(\frac{1}{z^2} + H_1(z)\right) = a_0 + a_1\left(\frac{1}{z^2} + H_1(z)\right) + a_2\left(\frac{1}{z^2} + H_1(z)\right)^2 + ... \tag{3.12}$$

Because R_n does not contribute in the $\frac{1}{z^2}$ term, we see that $a_1 = \frac{1}{n^2}$, and because the expression for $\wp(nz)$ contains no constant terms and no terms of degree less than 2, we see that $a_0 = a_2 = a_3 = 0$, so,

$$\begin{aligned}P_n(\wp(z)) &= \tilde{P_n}(\wp(z)) + R_n(\wp(z))\\ &= \frac{1}{n^2}\wp(z) + R_n(\wp(z)).\end{aligned} \tag{3.13}$$

□

Remark 3.4. From this we see that the rational function $P_n(\wp(z))$ is always of degree 1.

Lemma 3.5. *z_0 is a pole of $\wp(nz)$ if and only if $nz_0 \in \Gamma$, or $nz_0 = a\omega_1 + b\omega_2$ so $z_0 = \frac{a\omega_1+b\omega_2}{n}$ for some $a, b \in \mathbb{Z}$.*

Theorem 3.6. *$\wp(nz) = P_n(\wp(z))$ contains n^2 roots.*

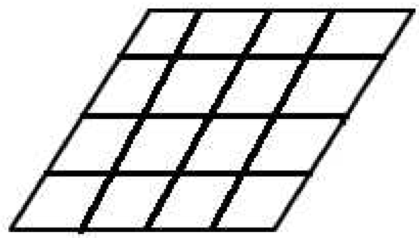

FIGURE 1. Grid Point Visualization

Proof. In the above lemma, clearly both a and b can take any integer value from 0 to $n-1$, so there are n^2 poles of $\wp(nz)$.

But as $\wp(nz) = P_n(\wp(z))$, then if a point z_0 is a pole of $\wp(nz)$, $\wp(z_0)$ must be a pole of $P_n(\wp(z))$, which means that $\wp(z_0)$ is a root of the denominator of the rational function $P_n(\wp(z))$. The one exception is the point $z_0 = 0$, because $\wp(0) = \infty$, which is not the root of any polynomial. Since $\wp(z)$ is well defined everywhere but on the lattice points, this means that the denominator of $P_n(\wp(z))$ must have $n^2 - 1$ roots, or must be of degree $n^2 - 1$. We proved earlier that $\deg P_n = 1$, so the numerator of $P_n(\wp(z))$ must be of degree n^2, so $\wp(nz) = P_n(\wp(z))$ contains n^2 roots. □

The grid point illustration in Figure 1 provides us with more information. First, it is easy to see that if n is composite, $\wp(nz)$ shares grid-point poles with $\wp(m_i z)$ where m_i are the factors of n. Our $\wp(4z)$ examples illustrates this easily; the halfway points are clearly also poles of $\wp(2z)$. For our polynomial $P_n(\wp(z))$, this means that the denominator of $P_n(\wp(z))$ contains all the factors in the denominator of $P_{m_i}(\wp(z))$.

We also know that $\wp(z)$ is even. On our grid this introduces a rotational symmetry about the center, where pairs of points yield the same value. In our $\wp(z)$ example, $\wp(\frac{3\omega_1+\omega_2}{4}) = \wp(-\frac{3\omega_1+\omega_2}{4}) = \wp(\frac{\omega_1+3\omega_2}{4})$

For the rational function $P_n(\wp(z))$, because two different grid points z_0 and $-z_0$ have the same value $\wp(z_0)$, this means that the value of $\wp(z_0)$ must be a double pole of the function $P_n(\wp(z))$. Thus, the denominator of the function factors into squared terms. The only exceptions to this are the halfway points $\frac{\omega_i}{2}$, where $i = 1, 2, 3$ and $\omega_3 = \omega_1 + \omega_2$. It is clear that $\frac{\omega_i}{2}$ and $-\frac{\omega_i}{2}$ are the same grid point, and so the value $\wp(\frac{\omega_i}{2})$ is only a single pole of $P_n(\wp(z))$. This may seem to contradict the fact that if $\frac{\omega_i}{2}$ is on our grid, then it is a double pole of $\wp(nz)$. The reason that this is not a contradiction comes from the fact that $\wp'(\frac{\omega_i}{2}) = 0$, as illustrated in the proof below.

Theorem 3.7. *If n is even, the halfway points $\frac{\omega_i}{2}$ are simple poles of $P_n(\wp(z))$, or equivalently,*

$$\lim_{z\to\frac{\omega_i}{2}} P_n(\wp(z))(\wp(z) - \wp(\frac{\omega_i}{2})) = c, \text{ for some } c \neq 0. \tag{3.14}$$

Proof. We know that if n is even, $\frac{\omega_i}{2}$ is a double pole of $\wp(nz)$, so we can expand as follows.

$$P_n(\wp(z)) = \wp(nz) = \frac{a_{-2}}{(nz - n\frac{\omega_i}{2})^2} + \frac{a_{-1}}{nz - n\frac{\omega_i}{2}} + \sum_{k=0}^{\infty} a_k (nz - n\frac{\omega_i}{2})^k, \tag{3.15}$$

hence

(3.16)

$$\lim_{z\to\frac{\omega_i}{2}} P_n(\wp(z))(\wp(z)-\wp(\frac{\omega_i}{2}) = \lim_{z\to\frac{\omega_i}{2}} \frac{a_{-2}(\wp(z)-\wp(\frac{\omega_i}{2}))}{n^2(z-\frac{\omega_i}{2})} + \frac{a_{-1}(\wp(z)-\wp(\frac{\omega_i}{2}))}{n(z-\frac{\omega_i}{2})}$$
$$+(\wp(z)-\wp(\frac{\omega_i}{2})\sum_{k=0}^{\infty} a_k(nz-n\frac{\omega_i}{2})^k.$$

The terms of the sum go to zero, and we may use the L'Hospital's Rule on the rational terms

$$= \lim_{z\to\frac{\omega_i}{2}} \frac{a_{-2}\wp'(z)}{2n^2(z-\frac{\omega_i}{2})} + \frac{a_{-1}\wp'(z)}{n}. \tag{3.17}$$

Because $\wp'(\frac{\omega_i}{2})=0$, the term on the right is 0, and we use the L'Hopital's Rule again on the term on the left,

$$= \lim_{z\to\frac{\omega_i}{2}} \frac{a_{-2}\wp''(z)}{2n^2} = \frac{a_{-2}\wp''(\frac{\omega_i}{2})}{2n^2} \neq 0. \tag{3.18}$$

□

Theorem 3.8. *Any collection of distinct $P_i(\wp(z))$'s are linearly independent.*

Proof. Note that if a grid point z_0 is not a halfway point $\frac{\omega_i}{2}$, then $\wp'(z_0)$ is not zero and we can use a similar grid argument as before to show that $\wp(z_0)$ is a double pole of $P_n(\wp(z))$. Furthermore because every distinct n gives at least some different grid points, we see that $P_n(\wp(z))$ must have different poles, and so for distinct n_i's, the $P_{n_i}(\wp(z))$'s are linearly independent. □

We now establish the main result of the paper.

Theorem 3.9. *In the expansion of $P_n(\wp(z))$ into a sum of partial fractions,*

$$P_n(\wp(z)) = \frac{c_{-2}}{(\wp(z)-\wp(z_0))^2} + \frac{c_{-1}}{\wp(z)-\wp(z_0)} + R_0(\wp(z)), \tag{3.19}$$

where $R_0(\wp(z))$ is some rational function, if z_0 is a pole, such that $z_0 \neq 0, \frac{\omega_i}{2}$, then

$$c_{-2} = \frac{\wp'(z_0)^2}{n^2} \text{ and } c_{-1} = \frac{\wp''(z_0)}{n^2}. \tag{3.20}$$

Proof. From the definition of $\wp(nz)$, we see that

$$\wp(nz) = \frac{1}{n^2(z-z_0)^2} - \frac{1}{n^2 z_0^2} + \sum_{\omega\in\Gamma,\omega\neq 0,nz_0} \left(\frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2}\right). \tag{3.21}$$

Now, we set $P_n(\wp(z))$ and $\wp(nz)$ equal to each other, and multiply the resulting expression by $(z-z_0)^2$

(3.22)

$$\frac{c_{-2}}{\frac{(\wp(z)-\wp(z_0))^2}{(z-z_0)^2}} + \frac{(z-z_0)c_{-1}}{\frac{\wp(z)-\wp(z_0)}{z-z_0}} + (z-z_0)^2 R_0(\wp(z)) = \frac{1}{n^2}$$
$$-(z-z_0)^2\left(\frac{1}{n^2z^2} - \frac{1}{n^2z_0^2} + \sum_{\omega\in\Gamma,\omega\neq 0,nz_0}\left(\frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2}\right)\right).$$

Taking the limit as $z \to z_0$ gives

$$\frac{c_{-2}}{\wp'(z_0)^2} = \frac{1}{n^2}. \tag{3.23}$$

So our result for c_{-2} follows.

Now, we attempt to find c_{-1}. Again, we set $P_n(\wp(z)) = \wp(nz)$, but this time we multiply through the expression by $(z - z_0)$ to get

$$\begin{aligned}
&\frac{c_{-2}}{\frac{(\wp(z)-\wp(z_0))^2}{z-z_0}} + \frac{c_{-1}}{\frac{\wp(z)-\wp(z_0)}{z-z_0}} + (z-z_0)R_0(\wp(z)) = \frac{1}{n^2(z-z_0)} \\
&+(z-z_0)\left(\frac{1}{n^2z^2} - \frac{1}{n^2z_0^2} + \sum_{\omega\in\Gamma,\omega\neq 0, nz_0}\left(\frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2}\right)\right).
\end{aligned} \tag{3.24}$$

Rearranging terms and substituting for c_{-2} gives

$$\begin{aligned}
&\frac{c_{-1}}{\frac{\wp(z)-\wp(z_0)}{z-z_0}} + (z-z_0)R_0(\wp(z)) = \frac{1}{n^2(z-z_0)} - \frac{\wp'(z_0)^2}{\frac{n^2(\wp(z)-\wp(z_0))^2}{z-z_0}} \\
&+(z-z_0)\left(\frac{1}{n^2z^2} - \frac{1}{n^2z_0^2} + \sum_{\omega\in\Gamma,\omega\neq 0, nz_0}\left(\frac{1}{(nz-\omega)^2} - \frac{1}{\omega^2}\right)\right).
\end{aligned} \tag{3.25}$$

Taking the limit as $z \to z_0$ yields

$$\frac{c_{-1}}{\wp'(z_0)} = \frac{1}{n^2}\lim_{z\to z_0}\left(\frac{1}{z-z_0} - \frac{\wp'(z_0)^2}{\frac{(\wp(z)-\wp(z_0))^2}{z-z_0}}\right). \tag{3.26}$$

Because $z_0 \neq 0$, we can substitute the Taylor series expansion for $\wp(z)$

$$\wp(z) = \wp(z_0) + \wp'(z_0)(z-z_0) + \frac{\wp''(z_0)}{2!}(z-z_0)^2 + ... \tag{3.27}$$

into our expression, giving

$$\begin{aligned}
\frac{n^2c_{-1}}{\wp'(z_0)} &= \lim_{z\to z_0}\left(\frac{1}{z-z_0} - \frac{\wp'(z_0)^2}{\frac{(\wp(z_0)+\wp'(z_0)(z-z_0)+\frac{\wp''(z_0)}{2!}(z-z_0)^2+...-\wp(z_0))^2}{z-z_0}}\right) \\
&= \lim_{z\to z_0}\left(\frac{1}{z-z_0} - \frac{\wp'(z_0)^2}{\frac{((z-z_0)(\wp'(z_0)+\frac{\wp''(z_0)}{2!}(z-z_0)+...))^2}{z-z_0}}\right) \\
&= \lim_{z\to z_0}\left(\frac{(\wp'(z_0)+\frac{\wp''(z_0)}{2!}(z-z_0)+...)^2 - \wp'(z_0)^2}{(z-z_0)(\wp'(z_0)+\frac{\wp''(z_0)}{2!}(z-z_0)+...)^2}\right) \\
&= \lim_{z\to z_0}\left(\frac{\wp'(z_0)^2 + 2(z-z_0)\wp'(z_0)\frac{\wp''(z_0)}{2!} + ... - \wp'(z_0)^2}{(z-z_0)(\wp'(z_0)+\frac{\wp'(z_0)}{2!}(z-z_0)+...)^2}\right) \\
&= \lim_{z\to z_0}\left(\frac{\wp'(z_0)\wp''(z_0)+...}{(\wp(z_0)+\frac{\wp''(z_0)}{2!}(z-z_0)+...)^2}\right) \\
&= \frac{\wp'(z_0)\wp''(z_0)}{\wp'(z_0)^2}.
\end{aligned}$$

□

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