Lecture 7:
Functions of a Complex Variable

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7.1 Notation and terminology

If S is a set of complex numbers and $f : S \to \mathbb{C}$ is defined for all $z \in S$, we call S the *domain* of f. Note that the domain of a function f need not be a domain. As is customary, when a domain is not specified for a function f, it is assumed to be the largest possible set for which f is defined.

Example 7.1. The domain of

$$f(z) = \frac{1}{z^2 + 1}$$

is

$$S = \{z \in \mathbb{C} : z \neq \pm i\}.$$

If $f : S \to \mathbb{C}$, then $f(z)$ is a complex number and so we may write

$$f(z) = u + iv$$

for real numbers u and v which depend on z. Now $z = x + iy$ for some real numbers x and y, so in fact u and v are functions of real numbers x and y. That is, we may write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions of two real variables. If instead we write z in polar coordinates, say, $z = re^{i\theta}$, then we may write

$$f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + iv(r, \theta).$$

Example 7.2. If $f(z) = z^3$, then

$$\begin{aligned} f(x+iy) &= (x+iy)^3 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3), \end{aligned}$$

so

$$u(x,y) = x^3 - 3xy^2$$

and

$$v(x,y) = 3x^2y - y^3.$$

In polar coordinates,

$$f\left(re^{i\theta}\right) = r^3(\cos(3\theta) + i\sin(3\theta)),$$

so

$$u(r,\theta) = r^3\cos(3\theta)$$

and

$$v(r,\theta) = r^3\sin(3\theta).$$

Definition 7.1. Given complex numbers a_0, a_1, ..., a_n, with $a_n \neq 0$, we call

$$P(z) = a_0 + a_1z + a_2z^2 \cdots + a_nz^n$$

a *polynomial* of *degree* n. If P and Q are polynomials, we call

$$R(z) = \frac{P(z)}{Q(z)}$$

a *rational function.*

In some instances we will need to consider a relationship which associates multiple values with a given input value, as illustrated in the following example.

Example 7.3. Recall that if $z = re^{i\theta} \in \mathbb{C}$, $z \neq 0$ and $-\pi < \theta \leq \pi$, then the two square roots of z are

$$c_0 = \sqrt{r}e^{i\frac{\theta}{2}}$$

and

$$c_1 = \sqrt{r}e^{i\left(\frac{\theta}{2}+\pi\right)}$$

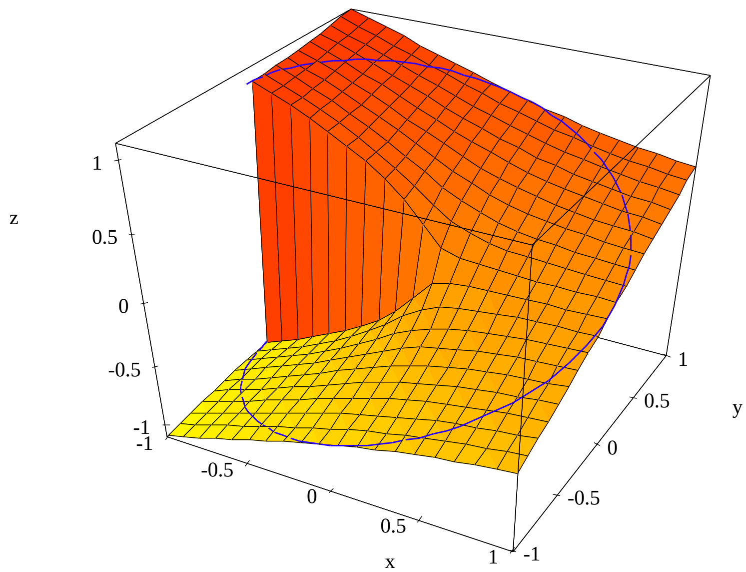

Figure 7.1: Plot of the imaginary part of the principal square root of z

$$
\begin{aligned}
&= \sqrt{r}\left(\cos\left(\frac{\theta}{2}+\pi\right)+i\sin\left(\frac{\theta}{2}+\pi\right)\right) \\
&= -\sqrt{r}\left(\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right) \\
&= -\sqrt{r}e^{i\frac{\theta}{2}} \\
&= -c_0.
\end{aligned}
$$

Hence

$$z^{\frac{1}{2}} = \pm\sqrt{r}e^{i\frac{\theta}{2}}.$$

We say that $z^{\frac{1}{2}}$ is a *multi-valued function*. We may create a function (that is, a *single-valued* function) $f : \mathbb{C} \to \mathbb{C}$ from this multi-valued function by defining $f(0) = 0$ and

$$f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$$

for $z = re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. The plot in Figure 7.1 displays the imaginary part of $f(z)$, the blue curve illustrating the curve on the surface above the unit circle. Note the split in the surface along the negative real axis.