Lecture 6: Some Topology

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6.1 Topological terminology

Definition 6.1. Given a complex number z_0 and a real number $\epsilon > 0$, we call the set

$$\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$$

an ϵ *neighborhood* of z_0 and the set

$$\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$$

a *deleted neighborhood* of z_0.

Definition 6.2. If $S \subset \mathbb{C}$ and $z_0 \in S$, we say z_0 is an *interior point* of S if for some $\epsilon > 0$ the ϵ neighborhood of z_0 lies in S. If $z_0 \notin S$, we say z_0 is an *exterior point* of S if for some $\epsilon > 0$ the ϵ neighborhood of z_0 has no points in common with S. We call a point z_0 which is neither an interior point nor an exterior a *boundary point* of S. We call the set of all boundary points of S the *boundary* of S, the set of all interior points of S the *interior* of S, and the set of all exterior points of S the *exterior* of S.

Example 6.1. Let

$$S = \{z \in \mathbb{C} : 1 < |z| \leq 2\}.$$

The interior of S is

$$\{z \in \mathbb{C} : 1 < |z| < 2\},$$

the exterior is

$$\{z \in \mathbb{C} : |z| < 1 \text{ or } |z| > 2\},$$

and the boundary of S is

$$\{z \in \mathbb{C} : |z| = 1 \text{ or } |z| = 2\}.$$

Definition 6.3. We say a set S is open if it contains none of its boundary points and *closed* if contains all of its boundary points. The *closure* of a set S is the union of S with its boundary points.

Proposition 6.1. The closure of a set S is closed.

Proof. Let C be the closure of S and let z be a boundary point of C. We need to show that $z \in C$. If $z \in S$, then $z \in C$, so suppose $z \notin S$. If $z \notin C$, then z is not a boundary point of S, and so is in the exterior of S. Hence there exists an $\epsilon > 0$ such that the ϵ neighborhood of z, call it U, does not intersect S. However, since z is a boundary point of C, there exists a boundary point of S, say w, with $w \in U$. Let $\delta = |z - w|$ and let $\gamma = \epsilon - \delta$. The γ neighborhood of w is a subset of U, but must intersect S since w is a a boundary point of S. This contradicts the choice of ϵ, and so $z \in C$ and C is closed. $\square$

Example 6.2. If

$$S = \{z \in \mathbb{C} : 1 < |z| \leq 2\},$$

as in the previous example, then S is neither open nor closed. Moreover, the closure of S is

$$C = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\},$$

which is closed.

Example 6.3. $S = \{z \in \mathbb{C} : |z| < 1\}$ is open, while its closure $D = \{z \in \mathbb{C} : |z| \leq 1\}$ is closed.

Example 6.4. The set of all complex numbers $\mathbb{C}$ is both open and closed.

Definition 6.4. We say a set S is *connected* if for each pair of points z_1 and z_2 in S there exists a sequence of points $w_0, w_1, \ldots, w_n$ such that $w_0 = z_1$, $w_n = z_2$, and the line segment from w_{i-1} to w_i lies in S for $i = 1, 2, \ldots, n$.

Example 6.5. The annulus

$$S = \{z \in \mathbb{C} : 1 < |z| < 2\}$$

is connected, as is the disk

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The set

$$T = \{z \in \mathbb{C} : |z| < 1 \text{ or } |z - 2i| < 1\}$$

is not connected.

Definition 6.5. We call an open connected set a *domain* and a domain along with some, all, or none of its boundary points a *region*.

Example 6.6. The sets S and D in the previous example are both domains, and hence also regions. The closed disk

$$A = \{z \in \mathbb{C} : |z| \leq 1\}$$

is also a region.

Definition 6.6. We say a set S is *bounded* if for some $R > 0$

$$S \subset \{z \in \mathbb{C} : |z| < R\}.$$

If S is not bounded, we say S is *unbounded*.

Example 6.7. The disk $\{z \in \mathbb{C} : |z| < 1\}$ is bounded, whereas the set $\{z \in \mathbb{C} : |z| > 1\}$ is unbounded.

Definition 6.7. We call a point z_0 an *accumulation point* of a set S if each deleted neighborhood of S contains at least one point of S.

Note that if z_0 is an accumulation point of S and $z_0 \notin S$, then z_0 is a boundary point of S. Hence a closed set will contain each of it accumulation points. Conversely, if z_0 is a boundary point of a set S and $z_0 \notin S$, then z_0 is an accumulation point of S. Hence a set is closed if it contains each of its accumulation points.

Proposition 6.2. A set S is closed if and only if it contains all its accumulation points.

Example 6.8. Let

$$S = \left\{ \frac{1}{n} : n = 1, 2, 3, \ldots \right\}.$$

Then 0 is an accumulation point of S and also a boundary point of S. On the other hand, 1 is a boundary point of S, but not an accumulation point.

Example 6.9. Let $S = \{z \in \mathbb{C} : |z| < 1\}$. Then 0 is an accumulation point of of S, but not a boundary point. The points z for which $|z| = 1$ are both boundary points and accumulation points.