Lecture 44: Residues

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44.1 Some terminology

Recall that we say a point z_0 is a singular point of a function f if f is not analytic at z_0 but is analytic at some point in every neighborhood of z_0. We will say that z_0 is an *isolated* singular point if it is a singular point and there exists $\epsilon > 0$ such that f is analytic in the deleted neighborhood $0 < |z - z_0| < \epsilon$.

Example 44.1. Both $z = i$ and $z = -i$ are isolated singular points of

$$f(z) = \frac{1}{1 + z^2}.$$

Example 44.2. $z = 0$ is a singular point, but not an isolated singular point, of $f(z) = \mathrm{Log}(z)$.

If z_0 is an isolated singular point of f, then there exists $R > 0$ such that f is analytic in $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. It follows that $f(z)$ has a Laurent series representation for all $z \in D$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

In particular

$$b_1 = \frac{1}{2\pi i} \int_C f(z) dz,$$

where C is any positively oriented, simple closed contour which lies in D and has z_0 in its interior. In other words,

$$\int_C f(z)dz = 2\pi i b_1.$$

We call b_1 the *residue* of f at the isolated singular point z_0, which we will denote

$$\operatorname*{Res}_{z=z_0} f(z).$$

Example 44.3. Let C be the circle $|z-1| = 1$ and consider the integral

$$\int_C \frac{1}{(z+1)(z-1)^3} dz.$$

Now

$$\begin{aligned}
\frac{1}{z+1} &= \frac{1}{2-(-(z-1))} \\
&= \frac{1}{2} \cdot \frac{1}{1-\left(-\left(\frac{z-1}{2}\right)\right)} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (z-1)^n}{2^n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^n
\end{aligned}$$

for all z with $|z-1| < 2$, and so

$$\begin{aligned}
\frac{1}{(z+1)(z-1)^3} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^{n-3} \\
&= \frac{1}{2(z-1)^3} - \frac{1}{4(z-1)^2} + \frac{1}{8(z-1)} + -\frac{1}{16} + \\
&\quad + \frac{1}{32}(z-1) - \frac{1}{64}(z-1)^2 + \cdots
\end{aligned}$$

for all z with $0 < |z-1| < 2$. Hence

$$\operatorname*{Res}_{z=1} \frac{1}{(z+1)(z-1)^3} = \frac{1}{8},$$

and so

$$\int_C \frac{1}{(z+1)(z-1)^3} dz = \frac{\pi}{4} i.$$

Another approach to evaluating this integral begins with finding the partial fraction decomposition of

$$\frac{1}{(z+1)(z-1)^3}.$$

That is, there are constants A, B, C, and D such that

$$\begin{aligned}\frac{1}{(z+1)(z-1)^3} &= \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2} + \frac{D}{(z-1)^3} \\ &= \frac{A(z-1)^3 + B(z+1)(z-1)^2 + C(z+1)(z-1) + D(z+1)}{(z+1)(z-1)^3},\end{aligned}$$

which implies that

$$1 = A(z-1)^3 + B(z+1)(z-1)^2 + C(z+1)(z-1) + D(z+1).$$

Evaluating at $z = 1$, we find that $D = \frac{1}{2}$ and at $z = -1$ that $A = -\frac{1}{8}$. Evaluating at $z = 2$ then gives

$$1 = -\frac{1}{8} + 3B + 3C + \frac{3}{2}$$

and at $z = -2$ gives

$$1 = \frac{27}{8} - 9B + 3C - \frac{1}{2}.$$

Thus

$$\begin{aligned} 3B + 3C &= -\frac{3}{8} \\ -9B + 3C &= -\frac{15}{8}. \end{aligned}$$

from which it follows that $B = \frac{1}{8}$ and $C = -\frac{1}{4}$. Hence

$$\frac{1}{(z+1)(z-1)^3} = -\frac{1}{8(z+1)} + \frac{1}{8(z-1)} - \frac{1}{4(z-1)^2} + \frac{1}{2(z-1)^3}.$$

Since the term $-\frac{1}{8(z+1)}$ is analytic at $z = 1$, its contribution to the Laurent series at $z = 1$ will have only positive powers of $z - 1$; it follows that the remaining three terms contribute all the negative powers of $z-1$ to the Laurent series. Thus we see once again that

$$\operatorname*{Res}_{z=1} \frac{1}{(z+1)(z-1)^3} = \frac{1}{8}.$$

Alternatively, with the partial fraction decomposition we may observe that

$$\begin{aligned}
\int_C \frac{1}{(z+1)(z-1)^3} dz &= -\int_C \frac{1}{8(z+1)} dz + \int_C \frac{1}{8(z-1)} dz - \int_C \frac{1}{4(z-1)^2} \\
&\quad + \frac{1}{2(z-1)^3} dz \\
&= \frac{1}{8} \int_C \frac{1}{z-1} dz \\
&= \frac{1}{8} \cdot 2\pi i \\
&= \frac{\pi}{4} i.
\end{aligned}$$