Lecture 43: Multiplication and Division of Power Series

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43.1 Multiplication of power series

The following generalization of the power rule is known as *Leibniz's rule.*

Theorem 43.1. If f and g are n times differentiable at z, then

$$\frac{d^n}{dz^n} f(z)g(z) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z).$$

Proof. When $n = 1$, the result is just the product rule:

$$\frac{d}{dz} f(z)g(z) = f(z)g'(z) + f'(z)g(z).$$

Assuming the result is true for $n \geq 1$, we have

$$\begin{aligned}
\frac{d^{n+1}}{dz^{n+1}} f(z)g(z) &= \frac{d}{dz} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z) \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)}(z) g^{(n-k+1)}(z) + f^{(k+1)}(z) g^{(n-k)}(z) \right) \\
&= f(z)g^{(n+1)}(z) + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) f^{(k)}(z) g^{n-k+1}(z) \\
&\qquad + f^{(n+1)}(z)g(z)
\end{aligned}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(z) g^{n-k+1}(z),$$

which is the result for the $(n+1)$st derivative. $\square$

Now suppose

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

for all z in the open disk $D = \{z \in \mathbb{C} : |z - z_0| < R\}$. Then $f(z)g(z)$ is analytic in D, and so has a Taylor series representation

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for all z in D, where

$$\begin{aligned} c_n &= \frac{1}{n!} \frac{d^n}{dz^n} \bigg|_{z=z_0} f(z)g(z) \\ &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} \\ &= \sum_{k=0}^{n} a_k b_{n-k}. \end{aligned}$$

Note that this is exactly what we would obtain by formally multiplying the two series out term by term, as we would polynomials.

Example 43.1. We have

$$\frac{e^z}{1-z} = \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right)\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$
\begin{aligned}
&= 1 + (1+1)z + \left(\frac{1}{2} + 1 + 1\right) z^2 + \left(\frac{1}{6} + \frac{1}{2} + 1 + 1\right) z^3 + \cdots \\
&= 1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + \cdots
\end{aligned}
$$

for all z with $|z| < 1$.

Example 43.2. Since

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

for all $z \in \mathbb{C}$, we have

$$
\begin{aligned}
\sin^2(z) &= z^2 - \left(\frac{1}{6} + \frac{1}{6}\right) z^4 + \left(\frac{1}{120} + \frac{1}{36} + \frac{1}{120}\right) x^6 \\
&\quad - \left(\frac{1}{5040} + \frac{1}{720} + \frac{1}{720} + \frac{1}{5040}\right) x^8 + \cdots \\
&= z^2 - \frac{z^4}{3} + \frac{2}{45}z^6 - \frac{1}{315}x^8 + \cdots
\end{aligned}
$$

43.2 Division of power series

Division of power series may also be performed term by term as one would with polynomials.

Example 43.3. Since

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

for all z, we have, using term by term division,

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)} = \frac{1}{z} - \frac{1}{6}z + \frac{7}{360}z^3 - \frac{31}{15120}z^5 + \cdots$$

for all z with $0 < |z| < \pi$ (since $\sinh(z) = 0$ when $z = n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$).