Lecture 42:
Uniqueness of Series Representations

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42.1 Uniqueness of Taylor series

Theorem 42.1. If

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all z in an open disk $D = \{z \in \mathbb{C} : |z - z_0| < R\}$, then

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for $n = 0, 1, 2, \ldots$.

Proof. Let C be the circle $|z - z_0| = R_1$, where $0 < R_1 < R$, and let

$$g_n(z) = \frac{1}{2\pi i (z - z_0)^{n+1}},$$

$n = 0, 1, 2, \ldots$. Then

$$\int_C g_n(z) f(z) dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!}.$$

However, we also have

$$\int_C g_n(z) f(z) dz = \sum_{m=0}^{\infty} a_m \cdot \frac{1}{2\pi i} \int_C \frac{1}{(z - z_0)^{n-m+1}} dz.$$

Since

$$\int_C \frac{1}{(z-z_0)^{n-m+1}}dz = \begin{cases} 0, & \text{if } m \neq n, \\ 2\pi i, & \text{if } m = n, \end{cases}$$

it follows that

$$\frac{f^{(n)}(z_0)}{n!} = \int_C g_n(z)f(z)dz = a_n.$$

□

42.2 Uniqueness of Laurent series

Theorem 42.2. If

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

for all z in an open annulus $R_0 < |z-z_0| < R_1$, $R_0 \geq 0$, then

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

where C is any closed contour in the annulus with z_0 in its interior and $n = 0, \pm 1, \pm 2, \ldots$.

Proof. Similar to the previous proof, let

$$g_n(z) = \frac{1}{2\pi i(z-z_0)^{n+1}},$$

$n = 0, \pm 1, \pm 2, \ldots$. Then

$$\int_C g_n(z)f(z)dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

However, we also have

$$\int_C g_n(z)f(z)dz = \sum_{m=-\infty}^{\infty} a_m \cdot \frac{1}{2\pi i}\int_C \frac{1}{(z-z_0)^{n-m+1}}dz.$$

Since

$$\int_C \frac{1}{(z-z_0)^{n-m+1}}dz = \begin{cases} 0, & \text{if } m \neq n, \\ 2\pi i, & \text{if } m = n, \end{cases}$$

it follows that

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

$\square$