Lecture 41:
Integration of Power Series

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Mathematics 39

May 18, 2004

41.1 Integration of power series

Theorem 41.1. Suppose the power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has radius of convergence $R > 0$ and let $D = \{z \in \mathbb{C} : |z| < R\}$. If C is a contour in D and $g(z)$ is continuous on C, then

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz.$$

Proof. For any positive integer N, let

$$\rho_N(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n.$$

Then

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n g(z)(z - z_0)^n + g(z)\rho_N(z),$$

and so

$$\int_C g(z)S(z)dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz + \int_C g(z)\rho_N(z)dz.$$

We need to show that

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)dz = 0.$$

Let $M > 0$ be an upper bound for $|g(z)|$ on C and let L be the length of C. Given $\epsilon > 0$, let n_0 be a positive integer such that

$$|\rho_N(z)| < \frac{\epsilon}{ML}$$

for all $N > n_0$ and z on C. Then, for $N > n_0$,

$$\left| \int_C g(z)\rho_N(z)dz \right| < M \cdot \frac{\epsilon}{ML} \cdot L = \epsilon.$$

Thus

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)dz = 0.$$

Hence

$$\int_C g(z)S(z)dz = \lim_{N\to\infty} \sum_{n=0}^{N-1} a_n \int_C g(z)(z-z_0)^n dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z-z_0)^n dz.$$

□

Note that it now follows that if C is any closed contour in D, then

$$\int_C S(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz = 0.$$

Hence, by Morera's theorem, $S(z)$ is analytic on D.

Corollary 41.1. If $R > 0$ is the radius of convergence of the power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

then $S(z)$ is analytic in $D = \{z \in \mathbb{C} : |z| < R\}$.

Note that this now implies that if f is analytic at z_0 and z_1 is the point closest to z_0 at which f is not analytic, then $|z_1 - z_0|$ is the radius of convergence of the Taylor series for f at z_0.

Example 41.1. Let

$$f(z) = \begin{cases} \dfrac{\sin(z)}{z}, & \text{if } z \neq 0, \\ \\ 1, & \text{if } z = 0. \end{cases}$$

Now

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

for all $z \in \mathbb{C}$, so

$$\frac{\sin(z)}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

for all $z \neq 0$. Since the series on the right is 1 at $z = 0$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

for all $z \in \mathbb{C}$. Thus f is an entire function. In particular, f is continuous at $z = 0$, and so we have

$$1 = \lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin(z)}{z}.$$

41.2 Differentiation of power series

Theorem 41.2. Suppose the power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has radius of convergence $R > 0$ and let $D = \{z \in \mathbb{C} : |z| < R\}$. Then, for every $z \in D$,

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Proof. Let C be the circle $|w - z_0| = R_0$, where $|z - z_0| < R_0 < R$. Then

$$\begin{aligned} S'(z) &= \frac{1}{2\pi i} \int_C \frac{S(s)}{(s-z)^2} ds \\ &= \sum_{n=0}^{\infty} a_n \cdot \frac{1}{2\pi i} \int_C \frac{(s-z_0)^n}{(s-z)^2} ds \\ &= \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \end{aligned}$$

□

Example 41.2. We have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n n z^{n-1} &= \frac{d}{dz} \sum_{n=1}^{\infty} (-1)^n z^n \\ &= \frac{d}{dz} \left(\frac{1}{1+z} \right) \\ &= -\frac{1}{(1+z)^2}. \end{aligned}$$

Put another way,

$$\frac{1}{(1+z)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} n z^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1) z^n = 1 - 2z + 3z^2 - 4z^3 + \ldots.$$