Lecture 40: Continuity of Power Series

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40.1 Continuity

Theorem 40.1. Suppose the power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)$$

has radius of convergence R and let $D = \{z \in \mathbb{C} : |z - z_0| < R\}$. If, for $z \in D$, we let

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

then S is continuous on D.

Proof. Let $z \in D$. We need to show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(w) - S(z)| < \epsilon$$

whenever $|w - z| < \delta$. Choose a positive real number R_0 such that $|z - z_0| < R_0 < R$ and, for any positive integer N, let

$$S_N(w) = \sum_{n=0}^{N-1} a_n(w - z_0)^n$$

and

$$\rho_N(w) = \sum_{n=N}^{\infty} a_n(w - z_0)^n.$$

Since the power series converges uniformly on the closed disk $|w - z_0| \leq R_0$, we may choose a positive integer N_ϵ such that

$$|\rho_{N_\epsilon}(w)| < \frac{\epsilon}{3}.$$

for all w with $|w - z_0| \leq R_0$. Since $S_{N_\epsilon}(z)$ is continuous (since it is a polynomial), we may choose a $\delta_1 > 0$ such that

$$|S_{N_\epsilon}(w) - S_{N_\epsilon}(z)| < \frac{\epsilon}{3}$$

whenever $|w - z| < \delta_1$. Let δ be the smaller of δ_1 and $R_0 - |z - z_0|$. Then, for all w with $|w - z| < \delta$, we have

$$\begin{aligned}
|S(w) - S(z)| &= |(S_{N_\epsilon}(w) + \rho_{N_\epsilon}(w)) - (S_{N_\epsilon}(z) + \rho_{N_\epsilon}(z))| \\
&\leq |S_{N_\epsilon}(w) - S_{N_\epsilon}(z)| + |\rho_{N_\epsilon}(w)| + |\rho_{N_\epsilon}(z)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

□

40.2 Series with negative powers

Now suppose a series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

converges at a point $z_1 \neq z_0$. Let $w_1 = \frac{1}{z_1 - z_0}$. Then the series

$$\sum_{n=1}^{\infty} b_n w_1^n$$

converges, and so the power series

$$\sum_{n=1}^{\infty} b_n w^n$$

converges absolutely for all w with $|w| < |w_1|$. If we let $w = \frac{1}{z-z_0}$, this says that

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges absolutely whenever

$$\frac{1}{|z-z_0|} < \frac{1}{|z_1-z_0|},$$

that is, whenever $|z-z_0| > R_1$, where $R_1 = |z_1 - z_0|$. Hence

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges absolutely for all z in the exterior of the circle $|z - z_0| = R_1$. Moreover, the function to which the series converges is continuous.

More generally, one may show that if the series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges on an annulus $R_1 < |z-z_0| < R_2$, then, for any $R_1 < \rho_1 < \rho_2 < R_2$, both series converge uniformly on the closed annulus $\rho_1 \leq |z-z_0| \leq \rho_2$.