Lecture 4:
Polar Coordinates

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4.1 Polar coordinates

Recall: If (x, y) is a point in the plane, $(x, y) \neq (0, 0)$, r is the distance from (x, y) to the origin, and θ is the angle between the x-axis and the line passing through (x, y) and the origin (measured in the counterclockwise direction), then we call r and θ the *polar coordinates* of (x, y). Also recall that

$$r = \sqrt{x^2 + y^2},$$

$$\tan(\theta) = \frac{y}{x} \text{ (provided } x \neq 0\text{)},$$

$$x = r\cos(\theta),$$

and

$$y = r\sin(\theta).$$

It follows that if $z = x + iy$ is a complex number and r and θ are the polar coordinates of (x, y), then

$$z = r(\cos(\theta) + i\sin(\theta)).$$

Example 4.1. If $z = 1 + i$, then $r = \sqrt{2}$ and we may take $\theta = \frac{\pi}{4}$. That is,

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

Note, however, that θ is not unique. In particular, we could have used $\theta = \frac{9\pi}{4}$ or $\theta = -\frac{7\pi}{4}$. In general, any of the values

$$\frac{\pi}{4} + 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \ldots,$$

would work.

We call a given value of the polar coordinate θ an *argument* of z and denote the set of all possible arguments of z by $\arg z$. We call the value θ of $\arg z$ for which $-\pi < \theta \leq \pi$ the *principal value* of $\arg z$ and denote it by $\operatorname{Arg} z$.

Example 4.2. We have

$$\arg(-2 - 2i) = -\frac{3\pi}{4} + 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \ldots,$$

and

$$\operatorname{Arg}(-2 - 2i) = -\frac{3\pi}{4}.$$

Example 4.3. $\operatorname{Arg}(-3) = \pi$.

4.2 Euler's formula

We will explain this more carefully later on, but for now we introduce the notation

$$e^{i\theta} = \exp(i\theta) = \cos(\theta) + i\sin(\theta)$$

in order to provide a compact way to denote complex numbers in polar form. That is, if z is a complex number with polar coordinates r and θ, we may write

$$z = re^{i\theta}.$$

We will see that this agrees with the exponentional function from real variable calculus, but for now we must remember that it is only notation.

Example 4.4. We may now write

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

and

$$-2 - 2i = 2\sqrt{2}e^{-i\frac{3\pi}{4}}.$$

Example 4.5. Note that $e^{i\pi} = -1$.

Example 4.6. For θ going from 0 to 2π,

$$z = 4e^{i\theta}$$

is a parametrization of the circle of radius 4 centered at the origin.

More generally, for a fixed complex number z_0 and real number R,

$$z = z_0 + Re^{i\theta},$$

$0 \le \theta \le 2\pi$, parametrizes a circle of radius R with center at z_0.

Proposition 4.1. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are two complex numbers, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and, if $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Proof. The first result follows from noting that

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) \\ &\qquad + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \end{aligned}$$

For the second, first note that

$$(e^{i\theta})^{-1} = \frac{1}{e^{i\theta}} = \frac{1}{e^{i\theta}} \frac{e^{-i\theta}}{e^{-i\theta}} = \frac{e^{-i\theta}}{e^0} = e^{-i\theta}.$$

It now follows that

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

□

As a consequence of this proposition,

$$\arg(z_1 z_2) = \arg z_1 + \arg(z_2),$$
$$\arg(z^{-1}) = -\arg(z),$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

Moreover, note that we now have a geometric interpretation of complex multiplication: Multiplying z by w rotates z by an angle $\operatorname{Arg} w$ and stretches the result by a factor of $|w|$.

Example 4.7. If $z = \dfrac{1-i}{2+2i}$, then

$$\arg z = \arg(1-i) - \arg(2+2i) = -\frac{\pi}{4} - \frac{\pi}{4} + 2n\pi = -\frac{\pi}{2} + 2n\pi,$$

$n = 0, \pm 1, \pm 2, \ldots$. Hence

$$\operatorname{Arg} z = -\frac{\pi}{2}.$$

Indeed,

$$z = \frac{1-i}{2+2i}\frac{2-2i}{2-2i} = -\frac{4i}{8} = -\frac{1}{2}i.$$

4.3 DeMoivre's formula

If $z = re^{i\theta}$ and n is a postive integer, then

$$z^n = \underbrace{re^{i\theta} \cdot re^{i\theta} \cdots re^{i\theta}}_{n \text{ times}} = r^n e^{in\theta}.$$

Using our results about reciprocals, the result also holds when n is a negative integer. Moreover, if we use the convention that $z^0 = 1$, then

$$z^0 = r^0(\cos(0) + i\sin(0)) = r^0 e^{i\cdot 0}.$$

Hence we have

$$z^n = r^n e^{in\theta} \text{ for } n = 0, \pm 1, \pm 2, \cdots.$$

In the particular case when $r = 1$ this gives us

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) \text{ for } n = 0, \pm 1, \pm 2, \cdots,$$

which we call *de Moivre's formula*.

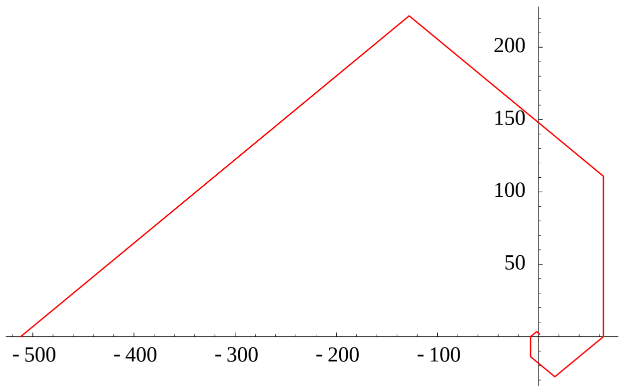

Figure 4.1: The points $(1+\sqrt{3}i)^n$ for $n = 1, 2, \ldots, 9$

Example 4.8. If $z = 1+\sqrt{3}i$, then $|z| = 2$ and $\operatorname{Arg} z = \frac{\pi}{3}$. Hence

$$z^9 = (1+\sqrt{3}i)^9 = \left(2e^{i\frac{\pi}{3}}\right)^9 = 2^9 e^{i3\pi} = -512.$$

Figure 4.1 displays z, z^2, $\ldots$, z^9.