Lecture 37: Laurent Series

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Mathematics 39

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37.1 Laurent's theorem

The following result is known as *Laurent's theorem*.

Theorem 37.1. Suppose $z_0 \in \mathbb{C}$, f is analytic in the domain

$$D = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\},$$

and C is any positively oriented, simple closed contour in D, with z_0 in the interior of C. Then, for any $z \in D$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

$n = 0, 1, 2, \ldots$, and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz,$$

$n = 1, 2, 3, \ldots$.

Note that we could write the series above as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}},$$

$n = 0, \pm 1, \pm 2, \ldots$. Moreover, note if f is analytic on the entire disk $|z-z_0| < R_2$, then $b_n = 0$ for all n and

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for $n = 0, 1, 2, \ldots$. That is, in this case Laurent's theorem reduces to Taylor's theorem.

Proof. We will assume $z_0 = 0$. The general case follows from a translation, as it did in the proof of Taylor's theorem. Choose $R_1 < r_1 < R_2$ and $r_1 < r_2 < R_2$ so that the annular region $r_1 < |w| < r_2$ contains both z and C. Let C_1 be the circle $|z| = r_1$ and let C_2 be the circle $|z| = r_2$. Let γ be a circle centered at z with radius smaller than both $r_2 - |z|$ and $|z| - r_1$. Give C_1, C_2, and γ positive orientations.

We now have, from extensions to the Cauchy-Goursat theorem, that

$$\int_{C_2} \frac{f(s)}{s-z}ds - \int_{C_1} \frac{f(s)}{s-z}ds - \int_{\gamma} \frac{f(s)}{s-z}ds = 0.$$

By the Cauchy integral formula,

$$\int_{\gamma} \frac{f(s)}{s-z}ds = 2\pi i f(z),$$

and so

$$\int_{C_2} \frac{f(s)}{s-z}ds - \int_{C_1} \frac{f(s)}{s-z}ds = 2\pi i f(z).$$

That is,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}ds - \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}ds \\ &= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}ds + \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{z-s}ds. \end{aligned}$$

We now proceed as we did in the proof of Taylor's theorem. We first note that, for any positive integer N,

$$\frac{1}{s-z} = \frac{1}{s}\frac{1}{1-\frac{z}{s}}$$

$$= \frac{1}{s}\left(\sum_{n=0}^{N-1}\frac{z^n}{s^n} + \frac{\frac{z^N}{s^N}}{1-\frac{z}{s}}\right)$$
$$= \sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N},$$

from which it follows that

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s-z}ds = \sum_{n=0}^{N-1}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}ds\right)z^n + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds$$
$$= \sum_{n=0}^{N-1}a_n z^n + \rho_N(z),$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}ds$$

and

$$\rho_N(z) = \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds.$$

Similarly, interchanging z and s, we see that, for any positive integer N,

$$\frac{1}{z-s} = \sum_{n=0}^{N-1}\frac{s^n}{z^{n+1}} + \frac{s^N}{(z-s)z^N} = \sum_{n=1}^{N}\frac{s^{n-1}}{z^n} + \frac{s^N}{(z-s)z^N},$$

from which it follows that

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{z-s}ds = \sum_{n=1}^{N}\left(\frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}ds\right)\frac{1}{z^n} + \frac{1}{2\pi i z^N}\int_{C_1}\frac{s^N f(s)}{z-s}ds$$
$$= \sum_{n=1}^{N}\frac{b_n}{z^n} + \sigma_N(z),$$

where

$$b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}ds$$

and

$$\sigma_N(z) = \frac{1}{2\pi i z^N}\int_{C_1}\frac{s^N f(s)}{z-s}ds.$$

Now let M_1 be the maximum value of $|f(s)|$ on C_1, let M_2 be the maximum value of $|f(s)|$ on C_2, and let $r = |z|$. Then, for s on C_2,

$$|s - z| \geq ||s| - |z|| = r_2 - r,$$

and so

$$\left| \frac{f(s)}{(s-z)s^N} \right| \leq \frac{M_2}{(r_2 - r)r_2^N}.$$

Hence

$$\begin{aligned} |\rho_N(z)| &= \left| \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds \right| \\ &\leq \frac{r^N}{2\pi} \cdot \frac{M_2}{(r_2 - r)r_2^N} \cdot 2\pi r_2 \\ &= \frac{M_2 r_2}{r_2 - r} \cdot \left(\frac{r}{r_2} \right)^N. \end{aligned}$$

Since $\frac{r}{r_2} < 1$,

$$\lim_{N \to \infty} \rho_N(z) = 0.$$

For s on C_2,

$$|z - s| \geq ||z| - |s|| = r - r_1,$$

and so

$$\left| \frac{s^N f(s)}{z - s} \right| \leq \frac{r_1^N M_1}{r - r_1}.$$

Hence

$$\begin{aligned} |\sigma_N(z)| &= \left| \frac{1}{2\pi i z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \right| \\ &\leq \frac{1}{2\pi r^N} \cdot \frac{r_1^N M_1}{r - r_1} \cdot 2\pi r_1 \\ &= \frac{M_1 r_1}{r - r_1} \cdot \left(\frac{r_1}{r} \right)^N. \end{aligned}$$

Since $\frac{r_1}{r} < 1$,

$$\lim_{N \to \infty} \sigma_N(z) = 0.$$

Thus we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

Finally, we note that, since f is analytic in D,

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(s)}{s^{n+1}}ds = \frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{n+1}}ds$$

for $n = 0, 1, 2, \ldots$, and

$$b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(s)}{s^{-n+1}}ds = \frac{1}{2\pi i}\int_{C}\frac{f(s)}{s^{-n+1}}ds$$

for $n = 1, 2, 3, \ldots$. This completes the proof when $z_0 = 0$. When $z_0 \neq 0$, define $g(z) = f(z + z_0)$ and proceed as in the proof of Taylor's theorem. $\square$