Lecture 35:
Taylor Series

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35.1 Taylor series

Definition 35.1. If f is analytic at a point $z_0 \in \mathbb{C}$, we call the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

the *Taylor series* of f about z_0. When $z_0 = 0$, we call

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

the *Maclaurin series* of f.

The following fundamental theorem is known as *Taylor's theorem*.

Theorem 35.1. If $R_0 > 0$, $z_0 \in \mathbb{C}$, and f is analytic in the disk

$$D = \{z \in \mathbb{C} : |z - z_0| < R_0\},$$

then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for all $z \in D$.

Proof. We first assume $z_0 = 0$. Let $z \in D$, let $r = |z|$, let $r < r_0 < R_0$, and let C_0 be the positively oriented circle of radius r_0 centered at the origin. By the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} ds.$$

Now

$$\begin{aligned} \frac{1}{s-z} &= \frac{1}{s} \cdot \frac{1}{1-\frac{z}{s}} \\ &= \frac{1}{s} \left(\sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{\left(\frac{z}{s}\right)^N}{1-\frac{z}{s}} \right) \\ &= \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}. \end{aligned}$$

Hence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} z^n \int_{C_0} \frac{f(s)}{s^{n+1}} ds + \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)s^N} ds \\ &= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z), \end{aligned}$$

where

$$\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)}{(s-z)s^N} ds.$$

It remains to show that $\lim_{N\to\infty} \rho_N(z) = 0$. Let M be the maximum value of $|f(s)|$ on C_0 and note that

$$|s-z| \geq ||s| - |z|| = r_0 - r.$$

Then

$$\left| \frac{f(s)}{(s-z)s^N} \right| \leq \frac{M}{(r_0-r)r_0^N},$$

and so

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0-r)r_0^N} \cdot 2\pi r_0 = \frac{Mr_0}{r_0-r} \left(\frac{r}{r_0}\right)^N.$$

Since $\frac{r}{r_0} < 1$, we have $\lim_{N\to\infty} \rho_N(z) = 0$.

Finally, if $z_0 \neq 0$, let $g(z) = f(z + z_0)$. Then g is analytic when

$$|(z + z_0) - z_0| < R_0,$$

that is, when $|z| < R_0$. Hence

$$f(z + z_0) = g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

when $|z| < R_0$. Thus if $|z - z_0| < R_0$,

$$f(z) = f((z - z_0) + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

□