Lecture 34: Sequences and Series

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Mathematics 39

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34.1 Sequences

Definition 34.1. We say an infinite sequence z_1, z_2, $\ldots$, z_n of complex numbers has a *limit* z if for every $\epsilon > 0$ there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon$$

whenever $n > n_0$, in which case we write

$$\lim_{n \to \infty} z_n = z$$

and we say the sequence *converges*. If a sequence does not converge, we say it *diverges*.

As with limits of functions, a sequence can have at most one limit. Moreover, if $z_n = x_n + iy_n$ and $z = x + iy$, where $x_n, y_n, x, y \in \mathbb{R}$, then

$$\lim_{n \to \infty} z_n = z$$

if and only if both

$$\lim_{n \to \infty} x_n = x$$

and

$$\lim_{n \to \infty} y_n = y.$$

The proofs of these results parallel the corresponding proof for limits of functions.

Example 34.1. Suppose

$$z_n = \frac{3}{n^2} + i\left(1 + \frac{1}{n^2}\right).$$

Then

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{3}{n^2} + i \lim_{n\to\infty} \left(1 + \frac{1}{n^2}\right) = i.$$

We could verify this limit from the definition as well by first noting that

$$|z_n - i| = \left|\frac{3}{n^2} + i\frac{1}{n^2}\right| = \frac{\sqrt{10}}{n^2},$$

and so, given $\epsilon > 0$, $|z_n - i| < \epsilon$ whenever

$$n > \frac{\sqrt[4]{10}}{\sqrt{\epsilon}}.$$

34.2 Series

Definition 34.2. Given an infinite sequence $z_1, z_2, z_3, \ldots$, let

$$S_N = z_1 + z_2 + \cdots + z_N.$$

We call the sequence $S_1, S_2, S_3, \ldots$ an *infinite series*, which we denote

$$\sum_{n=1}^{\infty} z_n.$$

We call S_N a *partial sum*. If S_N converges with $S = \lim_{N\to\infty} S_N$, then we say $\sum_{n=1}^{\infty} z_n$ *converges* and write

$$\sum_{n=1}^{\infty} z_n = S.$$

If S_N does not converge, we say $\sum_{n=1}^{\infty} z_n$ *diverges*.

Suppose $z_n = x_n + iy_n$ and $S = X + iY$. Then it follows from previous results that

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{i=1}^{\infty} y_n = Y.$$

Proposition 34.1. If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Proof. Let $S = \sum_{n=1}^{\infty} z_n$ and $S_N = \sum_{n=1}^{N} z_n$. Then

$$\lim_{N\to\infty} z_N = \lim_{N\to\infty} (S_N - S_{N-1}) = S - S = 0.$$

□

Definition 34.3. We say an infinite series

$$\sum_{n=1}^{\infty} z_n$$

is *absolutely convergent* if the infinite series

$$\sum_{n=1}^{\infty} |z_n|$$

converges.

Proposition 34.2. If $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then it is convergent.

Proof. Suppose $z_n = x_n + iy_n$ and

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$

converges. Since

$$|x_n| \leq \sqrt{x_n^2 + y_n^2}$$

and

$$|y_n| \leq \sqrt{x_n^2 + y_n^2},$$

it follows, by the comparison test, that both

$$\sum_{n=1}^{\infty} |x_n|$$

and

$$\sum_{n=1}^{\infty} |y_n|$$

converge. Hence, by a result from calculus, both $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge. Thus $\sum_{n=1}^{\infty} z_n$ converges. □

Definition 34.4. Given a complex numbers $a_0, a_1, a_2, \ldots$, and z_0, we call an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

a *power series*

Example 34.2. Let $z \in \mathbb{C}$ and consider the power series

$$\sum_{n=0}^{\infty} z^n.$$

If

$$S_N(z) = \sum_{n=0}^{N-1} z^n = 1 + z + z^2 + \cdots + z^{N-1},$$

then, from an earlier homework problem,

$$S_N(z) = \frac{1 - z^N}{1 - z},$$

when $z \neq 1$. Let

$$S(z) = \frac{1}{1 - z}.$$

If we let

$$\rho_N(z) = S(z) - S_N(z) = \frac{z^N}{1 - z},$$

then

$$|\rho_N(z)| = \frac{|z|^N}{|1 - z|}.$$

It follows that

$$\lim_{N \to \infty} |\rho_N(z)| = 0$$

if and only if $|z| < 1$. That is.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

if and only if $|z| < 1$. Put another way, the power series

$$1 + z + z^2 + \cdots$$

converges to $\frac{1}{1-z}$ for all z in the open disk $|z| < 1$ and for no other points in the plane.