Lecture 32: Liouville's Theorem

Dan Sloughter
Furman University
Mathematics 39

May 3, 2004

32.1 Liouville's Theorem

The following remarkable result is known as *Liouville's theorem*.

Theorem 32.1. If $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded, then $f(z)$ is constant throughout the plane.

The proof of Liouville's theorem follows easily from the following lemma.

Lemma 32.1. Let C_R be the circle $|z - z_0| = R$, $R > 0$, and suppose f is analytic on the region consisting of C_R and the points in its interior. If M_R is the maximum value of $|f(z)|$ on C_R, then, for $n = 1, 2, 3, \ldots$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! M_R}{R^n}.$$

Proof. Since

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

and

$$\left|\frac{f(z)}{(z - z_0)^{n+1}}\right| = \frac{|f(z)|}{|z - z_0|^{n+1}} \leq \frac{M_R}{R^{n+1}},$$

we have

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n! M_R}{R^n}.$$

□

We may now return to the proof of Liouville's theorem.

Proof. Suppose f is entire and $f(z) \leq M$ for all $z \in \mathbb{C}$. From the lemma, we have, for any $z_0 \in \mathbb{C}$ and any $R > 0$,

$$|f'(z_0)| \leq \frac{M_R}{R} \leq \frac{M}{R}.$$

Letting $R \to \infty$, we have $|f'(z_0)| = 0$, and hence $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$. Thus $f(z) = c$ for some constant c and all $z \in \mathbb{C}$. $\square$

32.2 Polynomials

Now consider a polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

with $a_n \neq 0$. Suppose there does not exists a $z_0 \in \mathbb{C}$ for which $P(z_0) = 0$. Let

$$f(z) = \frac{1}{P(z)}.$$

Then f is entire. Moreover, if $n \geq 1$,

$$\lim_{z \to \infty} f(z) = 0$$

since

$$\begin{aligned}
\lim_{z \to 0} f\left(\frac{1}{z}\right) &= \lim_{z \to 0} \frac{1}{P\left(\frac{1}{z}\right)} \\
&= \lim_{z \to 0} \frac{1}{a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n}} \\
&= \lim_{z \to 0} \frac{z^n}{a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \\
&= \frac{0}{a_n} \\
&= 0.
\end{aligned}$$

Hence there exists $R > 0$ such that $|f(z)| < 1$ whenever $|z| > R$. Since f is continuous on the closed disk $|z| \leq R$, there exists $M > 0$ such that $|f(z)| \leq M$ whenever $|z| \leq R$. It follows that f is bounded on $\mathbb{C}$. But then, by Liouville's theorem, f is a constant function, which is true only if $n = 0$. Hence we have proven the following fundamental theorem of algebra.

Theorem 32.2. If P is a polynomial of degree $n \geq 1$, then there exists at least one point $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.

Given a polynomial P of degree $n \geq 1$ and a point z_1 for which $P(z_1) = 0$, one may show that there exists a polynomial Q of degree $n - 1$ such that

$$P(z) = (z - z_1)Q(z).$$

Proceeding in this way, it now follows that there exist constants c and z_k, $k = 1, 2, 3, \ldots, n$, such that

$$P(z) = c(z - z_1)(z - z_2) \cdots (z - z_n).$$

This is the *Fundamental Theorem of Algebra*.

Corollary 32.1. Every algebraist needs an analyst.