Lecture 31:
Derivatives of Analytic Functions

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31.1 The derivative of an analytic function

Lemma 31.1. Suppose C is a positively oriented, simple closed contour and R is the region consisting of C and all points in the interior of C. If f is analytic in R, then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds$$

for any z in the interior of C.

Proof. Let d be the minimum value of $|s - z|$ for $s \in C$. Note that d exists because $|s - z|$ is a continuous function and C is a closed, bounded set. Consider Δz with $0 < |\Delta z| < d$. Now, by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds$$

and

$$f(z + \Delta z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z-\Delta z}ds,$$

and so

$$f(z + \Delta z) - f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)}ds.$$

Hence

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds = \frac{1}{2\pi i}\int_C \left(\frac{f(s)}{(s-z-\Delta z)(s-z)}-\frac{f(s)}{(s-z)^2}\right)ds$$
$$= \frac{1}{2\pi i}\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}ds.$$

For $s \in C$,

$$|s-z| \geq d$$

and

$$|s-z-\Delta z| \geq ||s-z|-|\Delta z|| \geq d-|\Delta z|.$$

It follows that if M is the maximum value of $|f(s)|$ on C and L is the length of C, then

$$\left|\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}ds\right| \leq \frac{M|\Delta z|}{(d-|\Delta z|)d^2}L.$$

Since

$$\lim_{\Delta z\to 0}\frac{M|\Delta z|}{(d-|\Delta z|)d^2}L = 0,$$

we have

$$\lim_{\Delta z\to 0}\left(\frac{f(z+\Delta z)-f(z)}{\Delta z}-\frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds\right) = 0;$$

that is,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds.$$

□

A similar argument shows that

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)}{(s-z)^3}ds.$$

Now if f is analytic at a point z_0, then there exists a δ neighborhood of z_0 in which f is analytic. Letting C be the circle $|z-z_0| = \frac{\delta}{2}$, the above formula shows that f' is analytic on the $\frac{\delta}{2}$ neighborhood of z_0, and so, in particular, analytic at z_0. Repeating this argument, we have the following theorem.

Theorem 31.1. If f is analytic at z_0, then $f^{(n)}(z_0)$ exists, and is analytic, for all positive integers n.

Corollary 31.1. If $f(x+iy) = u(x,y) + iv(x,y)$ is analytic at $z = x + iy$, then u and v have continuous partial derivatives of all orders.

The following theorem is known as *Morera's theorem.*

Theorem 31.2. If f is continuous on a domain D and

$$\int_C f(z)dz = 0$$

for every closed contour C in D, then f is analytic in D.

Proof. By an earlier result, we know there exists an analytic function F with $F'(z) = f(z)$ for all $z \in D$. Hence f is analytic. □

More generally, it may be shown that if f satisfies the conditions of the lemma, then, for $n = 1, 2, 3, \dots,$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} ds.$$

31.2 Examples

Example 31.1. Suppose C is the circle $|z| = 1$ with positive orientation. Then

$$\int_C \frac{e^{3z}}{z^4} dz = \frac{3^3 e^0 2\pi i}{3!} = 9\pi i.$$

Example 31.2. If z_0 is point in the interior of the positively oriented, simple closed contour C, then

$$\int_C \frac{1}{z - z_0} dz = 2\pi i,$$

and, for $n = 2, 3, 4, \dots$,

$$\int_C \frac{1}{(z - z_0)^n} dz = 0.$$

Example 31.3. Let C be the circle $|z - i| = 1$ with positive orientation. Then

$$\int_C \frac{1}{(z^2+1)^2} dz = \int_C \frac{\frac{1}{(z+i)^2}}{(z-i)^2} dz = 2\pi i(-2(i+i)^{-3}) = \frac{\pi}{2}.$$