Lecture 3:
Moduli and Conjugates

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3.1 The modulus of a complex number

Definition 3.1. For $z = x + iy \in \mathbb{C}$, we call

$$|z| = \sqrt{x^2 + y^2}$$

the *modulus*, or *absolute value*, of z.

Note that if $z = x + iy$ is real, that is, if $y = 0$, then

$$|z| = \sqrt{x^2} = |x|.$$

That is, the modulus of a real number is just the ordinary absolute value. Geometrically, $|z|$ is the distance between z and 0, the origin.

Example 3.1. If $z = 3 + i$, then $|z| = \sqrt{9 + 1} = \sqrt{10}$.

Note that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

is the distance between z_1 and z_2.

Example 3.2. It follows that the set

$$C = \{z \in \mathbb{C} : |z + 2 - i| = 5\}$$

is a circle of radius 5 with center at $-2 + i$.

Also note that, since

$$|z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2,$$

we have $(\text{Re } z)^2 \leq |z|^2$ and $(\text{Im } z)^2 \leq |z|^2$, and hence

$$|\text{Re } z| \leq |z| \text{ and } |\text{Im } z| \leq |z|.$$

3.2 Two inequalities

We may restate the familiar *triangle inequality* for points in the plane using the modulus of complex numbers:

Proposition 3.1. For any complex numbers z_1 and z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Note that, since $|-z| = |z|$,

$$|z_1| = |(z_1 + z_2) - z_2| \leq |z_1 + z_2| + |z_2|,$$

and so

$$|z_1 + z_2| \geq |z_1| - |z_2|.$$

Similarly,

$$|z_1 + z_2| \geq |z_2| - |z_1|.$$

Hence we have:

Proposition 3.2. For any complex numbers z_1 and z_2,

$$|z_1 + z_1| \geq ||z_1| - |z_2||.$$

Example 3.3. Suppose z lies on the circle $|z - i| = 1$. Then, for example,

$$|z - 1| = |(z - i) + (i - 1)| \leq |z - i| + |i - 1| = 1 + \sqrt{2}$$

and

$$|z - 1| = |(z - i) + (i - 1)| \geq ||z - i| - |i - 1|| = |1 - \sqrt{2}| = \sqrt{2} - 1.$$

3.3 Conjugates

Definition 3.2. If $z = x + iy \in \mathbb{C}$, we call

$$\bar{z} = x - iy$$

the *conjugate* of z.

Geometrically, $\bar{z}$ is the reflection of z about the real axis. Note that

$$\bar{\bar{z}} = z$$

and

$$|\bar{z}| = |z|.$$

It is easy to show that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$$
$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

and

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

Now if $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy) = 2x,$$

so

$$\text{Re } z = \frac{z + \bar{z}}{2}.$$

Similarly,

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy,$$

so

$$\text{Im } z = \frac{z - \bar{z}}{2i}.$$

We also see that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Proposition 3.3. For any complex numbers z_1 and z_2,

$$|z_1 z_2| = |z_1||z_2|$$

and, if $z_2 \neq 0$,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Proof. The result follows from the fact that

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2.$$

□

Example 3.4. For example, if $|z| < 1$, it follows that

$$|3z^2 + 4z - 2| \leq 3|z|^2 + 4|z| + 2 < 3 + 4 + 2 = 9.$$