Lecture 28: The Cauchy-Goursat Theorem

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Mathematics 39

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28.1 The Cauchy-Goursat Theorem

We say a simple closed contour is *positively oriented* if when traversing the curve the interior always lies to the left. For example, a circle oriented in the counterclockwise direction is positively oriented.

The following theorem was originally proved by Cauchy and later extended by Goursat.

Theorem 28.1. If R is the region consisting of a simple closed contour C and all points in its interior and $f : R \to \mathbb{C}$ is analytic in R, then

$$\int_C f(z)dz = 0.$$

We need some terminology and a lemma before proceeding with the proof of the theorem. Given a simple closed contour C, let R be the region consisting of C and all points in the interior of C. Since R is a bounded region, we may find real numbers a, b, c, and d such that $R \subset [a, b] \times [c, d]$. We may then divide this square into four, equal, smaller squares by bisecting the original square with a horizontal and a vertical line. Moreover, any smaller square obtained in this way may itself be subdivided into still smaller squares. After any finite number of subdivisions, we have a *cover of* R *by squares* consisting of those squares which lie entirely within R and those *partial squares* which are the intersection of a square with R.

Lemma 28.1. Let C be a positively oriented, simple closed contour and let R be the region consisting of C and all points in its interior. Given an analytic function $f : R \to \mathbb{C}$ and any $\epsilon > 0$, we may find a covering of R by squares, say T_1, T_2, ..., T_n, such that for every $j = 1, 2, \ldots, n$, there exists a fixed point $z_j \in T_j$ such that

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon$$

for every $z \in T_j$, $z \neq z_j$

Proof. Suppose it is not possible to find such a covering by squares. Then for any given covering by squares, there exists a square, or partial square, σ_0 for which no finite number of subdivisions will yield a covering by squares that satisfies the conclusion of the lemma. Hence, upon subdividing σ_0 into four squares, there exists a square, or partial square, σ_1 for which no finite number of subdivisions will yield a covering satisfying the conclusion of the lemma. Repeating this process, we create an infinite nested sequence of squares (or partial squares)

$$\sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_k \supset \cdots,$$

each containing points of R and none satisfying the conclusion of the lemma. It follows (see the homework) that there is a point

$$z_0 \in \bigcap_{k=0}^{\infty} \sigma_k;$$

moreover, since each square contains points of R, z_0 must be an accumulation point of R. Since R is closed, it follows that $z_0 \in R$. Now f is analytic at z_0, and so $f'(z_0)$ exists and there exists a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_j} - f'(z_0) \right| < \epsilon$$

whenever $|z - z_0| < \delta$. However, we may choose K large enough so that if $z \in \sigma_K$, $|z - z_0| < \delta$, contradicting our assumption that no subdivision of σ_K satisfies the condition of the lemma. □

We may now prove the Cauchy-Goursat theorem.

Proof. Let R be the region consisting of a positively oriented, simple closed contour C and all points in the interior of C, and let $f : R \to \mathbb{C}$ be analytic in R. Given $\epsilon > 0$, consider a covering by squares as described in the lemma. For $z \in T_j$, define

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j), & \text{if } z \neq z_j, \\ 0, & \text{if } z = z_j. \end{cases}$$

Note that δ_j is continuous on T_j and $|\delta_j(z)| < \epsilon$ for all $z \in T_j$. Let C_j be the positively oriented boundary of T_j and note that

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz$$

since the integrals along sides which are common to two squares cancel out.

Now for $z \in T_j$,

$$f(z) = f(z_j) + (z - z_j)(\delta_j(z) + f'(z_j)) = f(z_j) + f'(z_j)z - z_j f'(z_j) + (z - z_j)\delta_j(z).$$

Hence

$$\begin{aligned} \int_{C_j} f(z)dz &= f(z_j)\int_{C_j} dz - z_j f'(z_j)\int_{C_j} dz + f'(z_j)\int_{C_j} z dz \\ &\qquad + \int_{C_j} (z - z_j)\delta_j(z)dz \\ &= \int_{C_j} (z - z_j)\delta_j(z)dz. \end{aligned}$$

Thus

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz,$$

and so

$$\left| \int_C f(z)dz \right| \leq \sum_{j=1}^{n} \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right|.$$

Let s_j be the length of a side of T_j. Then, for $z \in C_j$,

$$|z - z_j| \leq \sqrt{2} s_j,$$

and so

$$|(z - z_j)\delta_j(z)| = |z - z_j||\delta_j(z)| < \sqrt{2}s_j\epsilon$$

for all $z \in C_j$.

Now let L be the length of C, L_j be the length of $C \cap T_j$, $A_j = s_j^2$, and S be the length of the side of the original square enclosing R. Then the length of C_j is less than or equal to

$$4s_j + L_j,$$

so

$$\begin{aligned}\left|\int_{C_j} (z - z_j)\delta_j(z)dz\right| &< (\sqrt{2}s_j\epsilon)(4s_j + L_j) \\ &= (4\sqrt{2}A_j + \sqrt{2}s_jL_j)\epsilon \\ &< (4\sqrt{2}A_j + \sqrt{2}SL_j)\epsilon.\end{aligned}$$

Hence

$$\sum_{j=1}^{n}\left|\int_{C_j} (z - z_j)\delta_j(z)dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\epsilon.$$

Thus we have shown that, no matter how small we choose $\epsilon > 0$, we have

$$\left|\int_C f(z)dz\right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\epsilon;$$

since S and L are fixed and ϵ may be made arbitrarily small, it follows that

$$\int_C f(z)dz = 0.$$

□